

Light scalars from a compact fifth dimension

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We consider a general five-dimensional sigma-model coupled to gravity, with any number of scalars and general sigma-model metric and potential. We discuss in detail the problem of the boundary conditions for the scalar fluctuations, in the case where the fifth dimension is compact, and provide a simple (and very general) algorithmic procedure for computing the spectrum of physical scalar fluctuations of the fully back-reacted system. Focusing in particular on the conditions under which the spectrum of scalar excitations (glueballs) contains parametrically light states, we apply the formalism to some especially simple toy models, which can be thought of as the gauge/gravity duals of strongly-coupled, non-conformal four-dimensional gauge theories. Our examples are chosen both within the context of phenomenological effective field theory constructions (bottom-up approach), and within the context of consistent truncations of ten-dimensional string theories in the supergravity limit (top-down approach). In one of the examples, a light dilaton is present in the spectrum in spite of the presence of a bad naked singularity in the deep IR, near which the RG flow of the dual theory is certainly very far away from any fixed point. If this feature were to persist in a complete model in which the singularity is resolved, this would prove that a light dilaton is to be expected in at least certain walking technicolor theories. We provide here all the technical details for testing this statement, once such a complete model is identified.

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I. INTRODUCTION

Many physical systems are described by strongly-coupled field theories, the dynamics of which is encoded in the fixed points of their renormalization group flow. Long-distance properties of such systems can be classified in terms of universal coefficients, which depend only on the properties of the system very close to the fixed points, but do not depend on the model-dependent features of the flows away from such fixed points. On the other hand, there are many cases in which the knowledge of the physics at the fixed points does not provide enough information as to allow one to compute phenomenologically important physical quantities that are experimentally measurable. In these latter cases, traditional field theory techniques are not powerful enough to yield robust predictions, which can be compared to the experimental data, mainly because of the strongly-coupled nature of the underlying dynamics.

One such example emerges in the context of dynamical electro-weak symmetry breaking, in particular in walking technicolor [1]. In this case, the underlying strong dynamics is quasi-conformal (approaching an IR fixed-point) over a range of energies above the electro-weak scale, but ultimately yields to confinement and to the formation of symmetry-breaking condensates at the electro-weak scale. One might expect that there exists a sense in which the condensates break spontaneously the (approximate) dilatation symmetry of the system near the (approximate) fixed point, thus leading to the appearance of a light dilaton (the pseudo-Goldstone boson of scale-invariance) in the spectrum of composite states. From a phenomenological point of view, this example is of most urgent importance, because such a light dilaton might mimic the properties of the Higgs particle of weakly-coupled models such as the minimal version of the Standard Model [2]. Unfortunately, because of the strong dynamics, and because the physics of a massive state such as the dilaton depends not only on the (universal) properties of the fixed-point, but also on the (non-universal) RG flow that yields confinement and chiral symmetry breaking itself, neither a firm confirmation nor a disproof of the existence of a light dilaton in walking technicolor has been provided by conventional field-theory methods, in spite of many attempts [3]. For recent work supporting the idea that such a light dilaton exists, see for instance [4, 5].

In recent years, the discovery of gauge/gravity dualities provided a new tool, that allows to reformulate field-theory problems emerging within four-dimensional strongly-coupled systems in terms of weakly-coupled extra-dimensional systems [6]. In its original formulation [7], the idea is to relate a particularly simple and symmetric 10-dimensional background ($AdS_5 \times S^5$) to a very special conformal four-dimensional theory ($\mathcal{N} = 4$ supersymmetric $SU(N)$ gauge theory), in the sense that a prescription is given for computing the generating functionals of correlation functions on the two sides of the correspondence, and the physical results agree. More recent developments provided large classes of dual models that correspond to non-conformal field theories with much less supersymmetry, such as those in which the 10-dimensional background is constructed starting from the conifold and its variations [8], and those that are related to controllable deformations of the $\mathcal{N} = 4$ field theory [9].

It turns out that, for several reasons, all the models that are of phenomenological interest share some very general properties. In particular, the 10-dimensional metric is always written in terms of a non-compact five-dimensional part (four directions of which are directly related to the dual four-dimensional space, with the fifth dimension related to the energy scale at which the dual theory is tested), and a compact (internal) five-dimensional space, the isometries of which are related to the internal global symmetries of the field theory. Formally, this means that it is often possible,

and very convenient, to study the 10-dimensional dynamics by first studying its five-dimensional reduction. One also implements a consistent truncation that reduces the number (and simplifies the action) of the resulting active fields.

The problem is thus reduced to finding a background solution of the classical equations of the truncated five-dimensional theory, and then use it to construct the lift to a background solution of the full 10-dimensional system. In practice, this means that what one has to solve, in order to determine the background, are the classical equations of a five-dimensional sigma-model of n scalars coupled to gravity. Since one usually wants to preserve the Lorentz structure of the four-dimensional space-time, one also assumes that the background depends only on the radial direction. With all of this, the complexity of the original problem of finding fully back-reacted 10-dimensional backgrounds is turned into the more treatable problem of solving a one-dimensional classical system.

One can even use this formalism dispensing with the original problem of finding a consistent truncation, by simply writing a five-dimensional phenomenological model that captures the most important features of the dynamics, postponing the problem of its completion (all we are going to say applies also in the fake supergravity context [10]). In this spirit, a vast literature of phenomenological models exists which aim at capturing the most important aspects of the gauge/gravity dualities without dealing with the technical difficulties of a complete string-theory construction. Relevant examples include the Randall-Sundrum model [11], in which the sigma-model consists just of a cosmological constant, the Goldberger-Wise mechanism [12] (see also [13]), in which the model contains only one scalar (see also [14]), and many applications, such as the Higgsless models [15], the AdS/QCD models [16], some composite-Higgs models [17] and the holographic technicolor models [18]. Recently, a similar strategy has been proposed also in order to study lower-dimensional condensed matter systems (see [19] for an introduction to the subject).

After reinstating (in the five-dimensional action) the dependence on the Minkowski directions, one can study the spectrum of classical fluctuations, which can be interpreted as the composite states of the dual theory (the glueballs, for instance). From this, one can finally access those very non-trivial properties of the strong dynamics that we started by describing in the beginning of this introduction, and ask hard questions such as whether a specific model yields a light dilaton in the spectrum. Yet, there are still two difficulties to overcome, before a final answer to these questions can be provided.

First of all, because the spectrum of massive states is not a universal property, one has to construct and study a variety of explicit models, possibly such that a lift to a complete 10-dimensional theory exists, and such that the dual field theory has all the properties required by phenomenology. In the specific context of walking technicolor, this program has recently been initiated, with some very encouraging results [5, 20, 21] (see also [22] for a more extensive discussion of what one might want to achieve along this line). Yet, at present we are nowhere close to having constructed the actual 10-dimensional dual of a phenomenologically relevant four-dimensional model, as a substantial amount of model-building is required in order to do so.

A second difficulty is of a more technical nature, and is the main subject of the present paper. When studying the spectrum and the properties of the fluctuations, a long preliminary work appears to be necessary (see for example [23]) — mainly because of the non-trivial mixing between fluctuations of the bulk scalars and the five-dimensional metric. One hence would need a formalism that is general and simple enough to correctly incorporate the relevant dynamics without having to analyze all the details on a model-by-model basis. In the case of a single scalar with trivial sigma model, this program has been addressed time ago by several collaborations (see for example [24]). Yet, the formalism developed by these authors needs to be extended far beyond the level needed for a single scalar, in order for it to apply to a realistic gravity dual of a strong dynamics.

In [25], a step towards a systematic resolution of this technical problem was taken. By introducing appropriate gauge-invariant combinations of the original fluctuations (see also [26] and [24]), which from the four-dimensional point of view correspond to scalars, vectors and tensors, it was shown that, given a completely general (two-derivatives) sigma-model with n scalars, and with a superpotential W (from which the 5d potential V can be derived), it is possible to algebraically manipulate the system of linear differential equations so as to rewrite it in a sigma-model covariant form and reduce it to a set of n second-order equations for the same number of physical fluctuations, from which the spectrum of the spin-0 sector of the theory can be derived. It was also shown in [27] that the formalism can easily be generalized to the case where the superpotential W is not known (or does not exist), in which case one needs to know the sigma-model metric and the potential.

This important formal tool works thanks to the fact that one can use the five-dimensional diffeomorphism invariance in order to remove some of the unphysical fluctuations. However, in practical applications one has to generalize this instrument further, so that it applies to the case where the radial direction is not infinite and, hence, boundary actions may need to be added. For example, in many cases an IR boundary is present because of an end-of-space in the geometry (which must be the case when discussing the dual of a confining gauge theory). Also, in the UV it is often necessary to work with a finite cutoff, for three possible generic reasons. It might be known that the dual field theory requires a UV completion above a given scale, so that the UV cutoff is actually physical. Retaining a UV cutoff is also necessary for technical reasons related to holographic renormalization [28]. And finally, it may be that the strongly-interacting dual theory is (weakly) coupled with an external (weakly-coupled) four-dimensional

sector, modeled by UV-boundary interactions. In all these three cases, the boundary terms are going to break the five-dimensional diffeomorphism invariance, and hence some caution has to be used when applying the gauge-invariant formalism.

In this paper, we provide the general form of the boundary conditions for the scalar fluctuations, both in the case in which the five-dimensional dynamics is known in terms of a superpotential, and when only the potential exists, without any restriction on the number of sigma-model scalars or on the sigma-model metric. We discuss the residual freedom in the form of the boundary conditions, in particular in relation to the light scalars in the spectrum, one (linear combination) of which may be interpreted as a dilaton, while any others correspond either to ordinary pseudo-Nambu-Goldstone bosons of approximate global symmetries of the dual theory, or are the result of accidental cancellations. We illustrate the formalism hence derived by applying it to a set of simple phenomenological examples, for which we provide both (approximate) analytical results for the spectrum, and (exact) numerical studies.

We do not treat here the problem of holographic renormalization, which we postpone to future work. In particular, we will always consider the models as defined on a compact fifth-dimension, away from any possible singularities. In this way, all the states are going to be physical. In practical examples, one has also to decide how to couple the model to possible external (weakly-coupled) systems, and how to take the limits in which the IR and UV boundaries are removed. We will briefly comment on these issues in due time, but we are not going to provide a systematic prescription for doing so.

A. The algorithm

The main purpose of the paper is to provide the reader with a simple algorithmic procedure for computing the spectrum of scalar excitations. Assuming that a five-dimensional model is of interest (irrespective of the fact that it is built either as a phenomenological model, or as the consistent truncation of a given supergravity, or superstring, or M-theory), one should go through the following steps.

- One must first ensure that the model can be written in the general form we provide, i. e. as a two-derivative, 5-dimensional action involving n real scalars coupled to gravity. We do not consider the case where higher-derivative terms are present, and ignore the possibility that higher-spin fields (such as gauge bosons or fermions, for instance) are relevant in determining the background.
- One must assume that two boundaries are present in the fifth dimension, representing the UV and IR cutoffs of the dual theory. These cutoffs may have a physical meaning, in which case all the results will explicitly depend on the dual scales. Or they may be thought of as regulators, in which case the physical results should be obtained by extrapolating the final results of the calculation to the actual physical case (typically, one wants to remove the UV cutoff completely, while the IR cutoff will approach the end-of-space). The presence of the boundaries means that boundary actions must be added. We provide the most general form of such boundary actions, subject to the limitation that we do not include terms that depend on the Minkowski four-momentum q^2 . As we will see, this form is very constrained, although we restrict ourselves to the quadratic order.
- One has to solve the system of background equations, and find a suitable background, with the ansatz that all the background functions depend only on the radial direction. We provide the general form of the bulk equations and their boundary conditions derived from the complete action. We do so both in the case in which the sigma-model is described in terms of a potential, but also in the case in which a superpotential description is known. The latter has the advantage that the background is completely determined by a set of first-order (coupled and non-linear) differential equations.
- One has to solve the linearized, second-order equations for the fluctuations around the background. The spectrum is determined by solutions that satisfy the boundary conditions both in the UV and in the IR. We provide the complete set of bulk equations [25], directly in the physical basis, written in terms of the background solution. And we provide the boundary conditions, the form of which depends again on the background functions, but also on a set of parameters that incorporate the residual freedom in the choice of boundary actions. These parameters should be chosen on the basis of physical principles, and are hence model-dependent. However, in the absence of symmetry reasons, there is only one such choice that ensures the absence of any fine-tuning in the physical results, and in this limit the boundary conditions depend, again, only on the functions determining the background, evaluated at the boundaries.

This procedure is very general, and since we write the fluctuation equations and boundary conditions already in terms of physical fields, no algebraic manipulations are needed. The reader who wants to apply this procedure can

directly write the final system in terms of n gauge-invariant fluctuations and solve it (in most of the cases, this must be done numerically). The formalism of [25] allows to write all the relevant equations in an elegant form that is fully covariant with respect not only to the space-time, but also the internal sigma-model geometry. One should be careful in correctly using all the covariant derivatives, which are determined both by the space-time and sigma-model connections: we will present all the relevant (heavy) notation in Section II.

Again, we must stress that the generality of the boundary terms for the fluctuations is limited by the fact that we do not include q^2 -dependent terms. These are model-dependent, and important in the context of holographic renormalization, when trying to take the UV cutoff to infinity, a problem that we postpone to future work.

B. Reader's guide

The paper is organized as follows. In Section II we summarize the basic formalism we use. This is a rather technical section, which may be skipped at first reading. However, all the material contained is necessary in order to correctly interpret and use the basic equations appearing in the paper, and we find it convenient to group together all the necessary definitions in one place. Also, we discuss here some subtleties emerging in the introduction of gauge-invariant variables in the presence of boundaries, which clarify and complete the literature on the subject.

In Section III we write the action of the sigma-model coupled to gravity, the equations determining the background, and the final differential equations and boundary conditions satisfied by the physical fluctuations. The derivation of these results is summarized in the appendices. All the relevant equations are written in terms of the sigma-model metric, the background fields, and the potential (and, when available, superpotential).

In Section IV we present three particularly interesting examples, and derive some analytical results. In Section V we apply the mid-point determinant method [25] to study these examples numerically. The main purpose of the examples is to illustrate the procedure, and hence we choose them to be particularly simple. However, it turns out that their physical interpretations are quite interesting, and that this set of exercises also provides some important insight into how the regulation procedure may work in non-trivial physical cases.

The first example is based on the same action used in the GW mechanism, and allows us to compare our results to the literature, but also to generalize the results and discuss many interesting subtleties that have been ignored in the past. In particular, we explicitly show that some freedom in the definition of the boundary terms results in the possible appearance of additional light scalars besides the dilaton, and that hence one has to exercise some caution in interpreting the results. We also show that a light dilaton is present (in great generality) not only when the scaling dimension Δ of the field-theory deformation encoded in the background is small ($\Delta \ll 1$), but also for any $\Delta \gtrsim 2$.

The second example is taken from a peculiarly simple five-dimensional model constructed by consistent truncation of type-IIB supergravity. The dual gauge theory has many properties that resemble those of a QCD-like theory, in the sense that the formation of a condensate in the IR takes the theory away from its fixed-point, presumably leading to confinement. Unfortunately, the model (studied here at zero temperature) suffers from the appearance of a naked singularity in the background, which limits its physical meaning. Yet, it is interesting to study what happens to the spectrum in the limit where the IR cutoff approaches the singularity. As we will see, the procedure adopted here yields a spectrum that, while distorted by the presence of the singularity, does not show any signs of pathologies, suggesting that the procedure that we follow removes some of the unpleasant features of the background at the singularity.

The third example is a phenomenological model yielding a background that can be interpreted in terms of the RG flow between a UV fixed-point and an IR fixed point. We study in some detail what happens to the spectrum by comparing several backgrounds that differ only by the value of the scale at which the transition from the proximity to one fixed point to the other takes place.

In Section VI we present a set of field-theory arguments aimed at explaining the results of the previous two sections. We elaborate on possible interpretations of the five-dimensional models in terms of dual, strongly-coupled theories, and derive some lessons about the physics of the three models we considered. These lessons extend to any model the examples somehow approximate, and are hence of general interest.

In Section VII we conclude, by summarizing the most important equations needed in the proposed algorithmic procedure, by commenting on the limitations and subtleties involved in using the algorithm itself, by summarizing briefly the physics lessons we learned, and finally by outlining some possible future applications.

II. FORMALISM

We introduce here the main definitions and conventions we use in the paper. We do not make explicit use of supergravity transformation and other supersymmetric properties. We start from the definition of the geometric properties of a sigma-model of n scalars coupled to gravity in five dimensions. Most of the notation and conventions

we use are taken from [25]. The notation is somewhat heavy, and hence we devote some time to explain it, explicitly showing all the definitions used in the whole paper. We then explicitly discuss the effect of gauge transformations and introduce the gauge invariant variables that will be used throughout the paper.

A. Geometry

All the equations we write make use of the geometric properties of the sigma-model, and are hence completely covariant. The space-time, sigma-model and background covariant derivatives are written in terms of the space-time and sigma-model metric and of the background fields.

We use the following conventions. Capital roman indices $M = 0, 1, 2, 3, 5$ are five-dimensional space-time indexes, while greek indexes $\mu = 0, 1, 2, 3$ are restricted to the 4-dimensional Minkowski slices of the space. In this way, we label the space-time coordinates as $x^M = (x^\mu, r)$, with r the radial (fifth) direction. Lower-case roman indexes $a = 1, \dots, n$ refer to the sigma-model (internal) space.

We write the five-dimensional metric g_{MN} with signature $-++++$. We define the five-dimensional connection as

$$\Gamma_{MN}^P \equiv \frac{1}{2}g^{PQ} \left(\partial_M g_{QN} + \partial_N g_{QM} - \partial_Q g_{MN} \right), \quad (1)$$

and hence the covariant derivatives are of the form

$$\nabla_M T_N^P \equiv \partial_M T_N^P + \Gamma_{MQ}^P T_N^Q - \Gamma_{MN}^Q T_Q^P, \quad (2)$$

for a $(1,1)$ -tensor, and analogous for other tensors, in such a way as to ensure compatibility with $\nabla_P g_{MN} = 0$. The Riemann tensor, Ricci tensor and Ricci scalars are defined, respectively, as

$$R_{MRN}{}^P \equiv \partial_R \Gamma_{MN}^P - \partial_M \Gamma_{RN}^P + \Gamma_{MN}^Q \Gamma_{RQ}^P - \Gamma_{MR}^Q \Gamma_{NQ}^P, \quad (3)$$

$$R_{MN} = R_{MRN}{}^R, \quad (4)$$

$$R \equiv g^{MN} R_{MN}. \quad (5)$$

One important fact that we will use in this paper is that we will assume the space-time to have one compact dimension. It is convenient to choose the coordinates in such a way that the radial direction $r_1 < r < r_2$ is compact, with the slices of space-time with constant r supporting a Minkowski four-dimensional metric. The presence of four-dimensional boundaries means that we will need to use the induced four-dimensional analogs of all of the above geometric objects, which we will label as $\tilde{g}_{\mu\nu}$, $\tilde{\nabla}_\mu$ and so on. The boundary terms are built starting from the ortho-normalized vector N^M , defined so that

$$g^{MN} N_N N_M = 1, \quad (6)$$

$$\tilde{g}_{MN} N^N = 0, \quad (7)$$

which implies that $\tilde{g}_{MN} = g_{MN} - N_M N_N$. The extrinsic curvature is defined from

$$K_{MN} \equiv \nabla_M N_N, \quad (8)$$

as the contraction with the bulk metric

$$K = g^{MN} K_{MN}. \quad (9)$$

The boundary actions will contain terms proportional to K .

The field content of the 5-dimension action comprises a set of real scalar fields that we label as Φ^a with $a = 1, \dots, n$. In a way that is analogous to the space-time metric, we indicate with G_{ab} the sigma-model metric, which ultimately encodes the geometric properties of the internal space spanned by the scalars. In the case where the five-dimensional system is obtained by consistently truncating some higher-dimensional theory, the structure of the sigma-model is determined unambiguously by the fact that the scalars parameterize some coset space, which is in general non-compact, and which emerges from the fact that the compactification of the internal space results in the breaking of some global symmetry of the underlying theory. We will keep the sigma-model structure as general as possible, hence not committing ourselves to any specific realization of this structure.

Sigma-model indexes are lowered and raised by the sigma-model metric G_{ab} and its inverse G^{ab} defined by $G^{ab} G_{bc} = \delta^a_c$. When unambiguous, we use lower indices to denote field derivatives with respect to the Φ^a , so that for example given a scalar function V we define

$$V_a \equiv \partial_a V \equiv \partial V / \partial \Phi^a. \quad (10)$$

The sigma model connection is given by

$$\mathcal{G}_{ab}^d \equiv \frac{1}{2} G^{dc} \left(\partial_a G_{cb} + \partial_b G_{ca} - \partial_c G_{ab} \right), \quad (11)$$

and the Riemann tensor with respect to the non-linear sigma-model metric is given by

$$\mathcal{R}_{bcd}^a \equiv \partial_c \mathcal{G}_{bd}^a - \partial_d \mathcal{G}_{bc}^a + \mathcal{G}_{ce}^a \mathcal{G}_{bd}^e - \mathcal{G}_{de}^a \mathcal{G}_{bc}^e. \quad (12)$$

Using the sigma-model connection, we define the sigma-model covariant derivative. It is convenient to introduce also another convention. When a sigma-model index is placed after a “|”, it means that the sigma-model covariant derivative with respect to G_{ab} should be taken, which is defined as acting on a $(1, 1)$ sigma-model tensor X_a^d by

$$X_{a|b}^d \equiv D_b X_a^d \equiv \partial_b X_a^d + \mathcal{G}_{cb}^d X_a^c - \mathcal{G}_{ab}^c X_c^d, \quad (13)$$

and analogous expressions for other tensors.

Finally, one needs to rewrite in a covariant form those objects that have indexes both on the space-time and on the sigma model, making them into generalized tensors. However, we do not need to write the general form of the covariant derivative for this case, because our theory is written only in terms of space-time scalars (that carry sigma-model indexes) and the metric (which is a sigma-model scalar), so that the only object we actually need is the background-covariant derivative \mathcal{D}_M defined for a five-dimensional scalar, sigma-model $(1, 0)$ -tensor \mathbf{a}^a , via

$$\mathcal{D}_M \mathbf{a}^a \equiv \partial_M \mathbf{a}^a + \mathcal{G}_{bc}^a \partial_M \bar{\Phi}^b \mathbf{a}^c, \quad (14)$$

where $\bar{\Phi}^b$ means that Φ^b is evaluated on the classical background (as is $\mathcal{G} = \mathcal{G}(\bar{\Phi}^a)$). Notice that in the following we will assume the background functions to depend only on the radial direction r , and hence only the fifth component of the background-covariant derivative has a non-trivial connection contribution, while the other components reduce to ordinary derivatives.

B. ADM formalism

We derive all the relevant equations using the ADM formalism, the basic idea being that we will rewrite the metric g_{MN} and the scalars Φ^a as a background function plus general fluctuations, and then decompose the metric in terms of four-dimensional tensors by slicing the space-time along the radial direction. We start by writing the metric in the form

$$g_{MN} = \begin{pmatrix} \tilde{g}_{\mu\nu} & \nu_\nu \\ \nu_\mu & \nu_\mu \nu^\mu + (1 + \nu)^2 \end{pmatrix}. \quad (15)$$

Because we singled out the radial direction as orthogonal to the boundaries, the normal vector is defined by $N_M = (0, (1 + \nu))$ and $N^M = (1 + \nu)^{-1}(-\nu_\mu, 1)$, so that $\tilde{g}^{MN} = \text{diag}\{\tilde{g}^{\mu\nu}, 0\}$ (notice that this tensor is not the inverse of \tilde{g}_{MN}).

We assume that the background metric satisfies the ansatz

$$ds^2 = e^{2A} \eta_{\mu\nu} dx^\mu dx^\nu + dr^2, \quad (16)$$

with $A = A(r)$. Similarly for the scalar, we assume that $\langle \Phi^a \rangle = \bar{\Phi}^a(r)$, so that the background depends on the radial direction r , but not on x^μ . We fluctuate the whole system by expanding the scalars as (using the exponential map)

$$\Phi^a = \exp_{\bar{\Phi}}(\varphi)^a \equiv \bar{\Phi}^a + \varphi^a - \frac{1}{2} \mathcal{G}_{bc}^a \varphi^b \varphi^c + \dots, \quad (17)$$

and the metric (to first order in the fluctuations) as

$$\tilde{g}_{\mu\nu} = e^{2A} (\eta_{\mu\nu} + h_{\mu\nu}), \quad (18)$$

with

$$h^\mu{}_\nu = h^{TT^\mu}{}_\nu + \partial^\mu \epsilon_\nu + \partial_\nu \epsilon^\mu + \frac{\partial^\mu \partial_\nu}{\square} H + \frac{1}{3} \delta^\mu{}_\nu h, \quad (19)$$

where $h^{TT^\mu}{}_\nu$ is traceless and transverse, and ϵ^μ is transverse. Altogether, we have the fluctuation variables $\{\varphi^a, \nu, \nu^\mu, h^{TT^\mu}{}_\nu, h, H, \epsilon^\mu\}$.¹

¹ Notice that, with some abuse of notation, we identify the fluctuations ν and ν^μ with the components of the metric in the ADM formalism.

C. Diffeomorphism invariance

Here we write explicitly the gauge transformations, and discuss the fact that the boundary actions restrict their general form compared to what is allowed in the bulk. The starting point is the five-dimensional diffeomorphisms

$$\delta x^M = -\xi^M, \quad (20)$$

which imply

$$\delta \Phi^a = \xi^M \partial_M \Phi^a, \quad (21)$$

$$\delta g_{MN} = \partial_M \xi^R g_{RN} + \partial_N \xi^R g_{MR} + \xi^R \partial_R g_{MN}. \quad (22)$$

To first-order in the fluctuations, this yields the gauge transformations for all the fluctuations

$$\begin{aligned} \delta \varphi^a &= \bar{\Phi}'^a \xi^r, \quad \delta \nu = \partial_r \xi^r, \quad \delta \nu^\mu = \partial^\mu \xi^r + e^{2A} \partial_r \xi^\mu, \quad \delta h^{TT^\mu}_\nu = 0, \\ \delta \epsilon^\mu &= \Pi^\mu_\nu \xi^\nu, \quad \delta H = 2 \partial_\mu \xi^\mu, \quad \delta h = 6 A' \xi^r, \end{aligned} \quad (23)$$

where we defined the projector $\Pi^\mu_\nu \equiv \delta^\mu_\nu - \frac{\partial^\mu \partial_\nu}{\square}$. In all of this, all the functions depend on the five coordinates x^M . The five-dimensional part of the action is going to be invariant under all of these transformations.

However, we do have four-dimensional boundary actions, the very existence of which is not compatible with all of the above. To be more specific, the action is still symmetric under these transformations for generic $\xi^\mu(x^\mu, r)$, but we have to specify how to treat the diffeomorphisms in the fifth direction. Because we will explicitly write the boundaries to support localized actions at the points $r = r_i$, with $i = 1, 2$, one must require²

$$\xi^r(x^\mu, r_i) = 0. \quad (24)$$

This observation plays an important role in the subsequent discussion about gauge-invariance and gauge-fixing.

D. Gauge-invariant formalism

Generalizing the notation of [25], we define the following variables

$$\begin{aligned} \mathfrak{a}^a &= \varphi^a - \frac{\bar{\Phi}'^a}{6A'} h, \\ \mathfrak{b} &= \nu - \frac{\partial_r(h/A')}{6}, \\ \mathfrak{c} &= e^{-2A} \partial_\mu \nu^\mu - \frac{e^{-2A} \square h}{6A'} - \frac{1}{2} \partial_r H, \\ \mathfrak{d}^\mu &= e^{-2A} \Pi^\mu_\nu \nu^\nu - \partial_r \epsilon^\mu, \\ \mathfrak{e}^\mu_\nu &= h^{TT^\mu}_\nu. \end{aligned} \quad (25)$$

These are a generalization of the Mukanov-Sasaki variable [29]. By inspection, one can verify that these new variables are 5d gauge invariant.

Before proceeding, let us go through a counting exercise. Besides the n scalars in the sigma-model, the off-shell degrees of freedom derived from the dimensional reduction of the five-dimensional fluctuations of g_{MN} comprise another 15 components for a total of $15 + n$ components. Counting in the basis of original fluctuations $\{\varphi^a, \nu, \nu^\mu, h^{TT^\mu}_\nu, h, H, \epsilon^\mu\}$ yields the same number $n + 1 + 4 + 5 + 1 + 1 + 3 = 15 + n$, as it should (we counted 3 for transverse vectors and 4 for generic vectors). The counting of the gauge-invariant variables $\{\mathfrak{a}^a, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}^\mu, \mathfrak{e}^\mu_\nu\}$, however, yields $n + 1 + 1 + 3 + 5 = n + 10$ off-shell components. The five extra components of the original fluctuations are pure gauge, corresponding to the diffeomorphisms ξ^M .

One needs to show explicitly that these gauge-invariant variables are physically equivalent to the original set of fluctuations, so that the full set of equations can be rewritten directly in this form, hence removing all the possibly

² This expression could be made covariant, if one wanted to manifestly show that one does not need to commit to a specific choice of coordinates.

spurious gauge artifacts, while retaining all the physical information. In the absence of the boundary action, this is straightforward, because one can make a choice of ξ^M such as to set $h = 0 = H = \epsilon^\mu$, and hence by simple counting one can see that the whole system can be rewritten in equivalent form in terms of the gauge invariant variables. This can be verified explicitly to hold for the bulk equations (which hence are the ones derived in [25]).

In our specific case, though, the boundaries restrict the gauge transformations allowed. One has to show that the thus constrained system can still be completely rewritten in terms of the variables in Eq. (25). In order to do so, we use a different strategy. First, we observe that because of the restriction on ξ^r in Eq. (24), at the boundaries one can define two independent 4d gauge invariant variables, which replace \mathbf{c} :

$$\mathbf{c}_1 \equiv -\frac{e^{-2A}\Box h}{6A'}, \quad (26)$$

$$\mathbf{c}_2 \equiv e^{-2A}\partial_\mu\nu^\mu - \frac{1}{2}\partial_r H, \quad (27)$$

so that $\mathbf{c} = \mathbf{c}_1 + \mathbf{c}_2$. The original system of fluctuations is certainly equivalent to the set $\{\mathbf{a}, \mathbf{b}, \mathbf{c}_1, \mathbf{c}_2, \mathbf{d}, \mathbf{e}\}$. Then, we have to show that the boundary conditions actually remove the extra degrees of freedom, hence proving that the presence of the boundary actions consistently restricts both the fluctuations and the gauge transformations, so that the whole system (bulk and boundaries) can be fully expressed in terms of the variables in Eq. (25).

More precisely, one can show that one of the boundary conditions can be written in a 4d gauge invariant form as

$$\mathbf{c}_2 \Big|_{r_i} = 0. \quad (28)$$

This, together with the fact that in the bulk the gauge freedom allows to always set $\mathbf{c} = \mathbf{c}_1$, means that \mathbf{c}_2 is actually not a physical degree of freedom, and it can be set to zero everywhere, hence allowing for the whole set of fluctuation equations and boundary conditions to be written purely in terms of the variables in Eq. (25). In the appendices, the boundary condition (28) is derived by first choosing a gauge transformation ξ^μ such that $\nu^\mu(x, r) = 0$ everywhere, then showing that at the boundaries $\partial_r H|_{r_i} = 0$. Finally, we can use the residual gauge ξ^r to set $\partial_r H = 0$ also in the bulk (together with \mathbf{c}_2).

Working in the gauge $\nu^\mu(x, r) = 0$, and making use of the boundary condition (28), there is a straightforward and natural one-to-one map between the fluctuations $\{\varphi^a, \nu, h, \epsilon^\mu, h^{TT^\mu}_\nu\}$ and the gauge invariant variables $\{\mathbf{a}^a, \mathbf{b}, \mathbf{c}, \mathbf{d}^\mu, \mathbf{e}^\mu_\nu\}$. The main advantage of the gauge-invariant formalism is that it allows to decouple the equations in a very simple way [25], hence rendering the calculation of the spectrum much easier. In practice, this means that the equations for \mathbf{b} and \mathbf{c} are algebraic equations relating them to the dynamical variables \mathbf{a}^a , while the equations for \mathbf{d}^μ and \mathbf{e}^μ_ν decouple. Hence, the spectrum of scalar fluctuations can be identified by solving a set of n second-order differential equations involving only the variables \mathbf{a}^a , subject to boundary conditions that, again, involve only the variables \mathbf{a}^a .

III. DYNAMICS

In this section we summarize all the important equations that one has to solve in order to study the spectrum of a given model.

A. The complete action

We are now ready to write explicitly the complete action. Our starting point is the general definition of the five-dimensional sigma-model coupled to gravity. We write the action as

$$\begin{aligned} \mathcal{S} \equiv \int d^4x dr \Big\{ & \sqrt{-g}\Theta \left[\frac{1}{4}R + \mathcal{L}_5(\Phi^a, \partial_M \Phi^a, g) \right] \\ & + \sqrt{-\tilde{g}}\delta(r - r_1) [c_K K + \mathcal{L}_1(\Phi^a, \partial_\mu \Phi^a, \tilde{g})] \\ & - \sqrt{-\tilde{g}}\delta(r - r_2) [c_K K + \mathcal{L}_2(\Phi^a, \partial_\mu \Phi^a, \tilde{g})] \Big\}, \end{aligned} \quad (29)$$

where R is the Ricci scalar, where K is the extrinsic curvature, where the coupling $c_K = -1/2$ is fixed by consistency and where \mathcal{L}_i are the sigma-model actions. The step function is defined by $\Theta \equiv \Theta(r - r_1) - \Theta(r - r_2)$.

We define the action of the matter fields in terms of the real scalar fields $\Phi^a = \Phi^a(x^\mu, r)$ as

$$\mathcal{L}_5 \equiv -\frac{1}{2}G_{ab}g^{MN}\partial_M\Phi^a\partial_N\Phi^b - V(\Phi^a), \quad (30)$$

$$\mathcal{L}_1 \equiv -\lambda_{(1)}(\Phi^a), \quad (31)$$

$$\mathcal{L}_2 \equiv -\lambda_{(2)}(\Phi^a). \quad (32)$$

Hence, we assume that no kinetic term is present at the boundaries, but only localized potential terms. We will provide the explicit forms of the $\lambda_{(i)}$ terms later. Notice the different sign with which they enter the complete action.

B. Background equations

The background is determined by solving the classical equations, assuming that all the functions defining the background depend only on the radial direction r , and not on the x^μ . We take the variation of the complete action in order to determine the equations of motion for $A(r)$ and $\bar{\Phi}(r)$.³ The equation of motion for the scalars is

$$\bar{\Phi}''^a + 4A'\bar{\Phi}'^a + \mathcal{G}_{bc}^a\bar{\Phi}'^b\bar{\Phi}'^c - V^a = 0, \quad (33)$$

while Einstein's equations read

$$\begin{aligned} 6A'^2 + 3A'' &= -G_{ab}\bar{\Phi}'^a\bar{\Phi}'^b - 2V, \\ 6A'^2 &= G_{ab}\bar{\Phi}'^a\bar{\Phi}'^b - 2V. \end{aligned} \quad (34)$$

The boundary conditions satisfied by the background, at the UV ($r = r_2$) and IR ($r = r_1$) are

$$\bar{\Phi}'^a \Big|_{r_i} = \lambda_{(i)}^a \Big|_{r_i} \equiv G^{ab}\partial_b\lambda_{(i)} \Big|_{r_i}, \quad (35)$$

$$A' \Big|_{r_i} = -\frac{2}{3}\lambda_{(i)} \Big|_{r_i}. \quad (36)$$

Thus we see that to quadratic order the localized potentials $\lambda_{(i)}(\Phi^a)$ are constrained to have the following form⁴

$$\begin{aligned} \lambda_{(1)} &= -\frac{3}{2}A' \Big|_{r_1} + G_{ab}\bar{\Phi}'^a \Big|_{r_1} (\Phi^b - \Phi_1^b) + \frac{1}{2}D_a D_b \lambda_{(1)} (\Phi^a - \Phi_1^a) (\Phi^b - \Phi_1^b), \\ \lambda_{(2)} &= -\frac{3}{2}A' \Big|_{r_2} + G_{ab}\bar{\Phi}'^a \Big|_{r_2} (\Phi^b - \Phi_2^b) + \frac{1}{2}D_a D_b \lambda_{(2)} (\Phi^a - \Phi_2^a) (\Phi^b - \Phi_2^b), \end{aligned} \quad (37)$$

where $\Phi_{1,2}^a \equiv \bar{\Phi}^a|_{r_{1,2}}$ are the values assumed by the scalars at the boundaries. In effect, this is equivalent to imposing the condition that the scalars have fixed boundary values Φ_i^a .

1. First-order formalism

Assuming that there exists a superpotential W such that the potential V can be rewritten as

$$V = \frac{1}{2}G^{ab}W_a W_b - \frac{4}{3}W^2, \quad (38)$$

³ See Appendix A for details. An alternative way is to take the variation of the action by linearizing in $\{h, \nu, \varphi^a\}$. The terms proportional to Θ then give us the equations of motion satisfied by the background in the bulk, whereas the terms proportional to $\delta = \partial_r \Theta$ give us the boundary conditions.

⁴ Note that the covariant derivatives of $\lambda_{(i)}$ arise due to expanding Φ^a according to the exponential map Eq. (17).

with $W_a = \partial_a W = \partial W / \partial \Phi^a$, then the system can be reduced to a set of $n + 1$ first-order equations

$$A' = -\frac{2}{3}W, \quad (39)$$

$$\bar{\Phi}'^a = G^{ab}W_b = W^a, \quad (40)$$

in the sense that all solutions to Eqs. (39)-(40) are also solutions to the second-order equations Eqs. (33)-(34). However, for these to provide solutions to the original system, also the boundary conditions must be satisfied, implying a set of constraints on the form of the localized potentials $\lambda_{(i)}$. The general form of the localized potentials must be

$$\lambda_{(1)} = W(\Phi_1) + W_c(\Phi_1)(\Phi^c - \Phi_1^c) + \frac{1}{2}D_d D_c \lambda_{(1)}(\Phi^c - \Phi_1^c)(\Phi^d - \Phi_1^d), \quad (41)$$

$$\lambda_{(2)} = W(\Phi_2) + W_c(\Phi_2)(\Phi^c - \Phi_2^c) + \frac{1}{2}D_d D_c \lambda_{(2)}(\Phi^c - \Phi_2^c)(\Phi^d - \Phi_2^d). \quad (42)$$

Before we move onto studying the spectrum of fluctuations, a brief comment is needed. One might, legitimately, wonder why we allow ourselves the freedom to add localized potential terms for the scalars, but not localized kinetic terms, and/or more general functions not only of the fields but also of their (four-dimensional) derivatives. The fact that we truncate $\lambda_{(i)}$ at the quadratic order is just due to the fact that we are interested here only in the equations for the background and in the linearized equations for the fluctuations, so that higher-order terms have no effect. On the other hand, while terms that depend explicitly on the four-momentum q^2 do not enter the background equations, they do enter into the boundary conditions for the fluctuations. However, their presence and structure in entangled with the problem of holographic renormalization, in the sense that in order to systematically identify what such terms are needed, one has to study the (model-dependent) structure of divergences in the two-point functions. This suggests using some caution when discussing the spectrum.

C. Boundary conditions for the fluctuations

As discussed in Section IID, the equations for the fluctuations of the scalars and the metric can be rewritten in terms of gauge-invariant fields, up to some subtleties that we will discuss later. Most important, we anticipate here that there are two possible ambiguities in this procedure, which is not well-defined for the (interrelated) cases when $\bar{\Phi}'^a = 0$ and/or when the four-dimensional momentum $\square = -K^2 = m^2 \equiv q^2$ vanishes. On the one hand, the procedure is rigorous even for the case in which these are infinitesimal, and on the other hand all the examples one might think of that have any phenomenological relevance will not contain exactly massless scalars, none of the global symmetries being exact. We will come back to these questions later in the next section.

With all of these caveats, the final result is that the spectrum can be obtained by solving the following second-order differential equation for a set of n gauge-invariant scalar fluctuations denoted by \mathbf{a}^a :

$$\left[\mathcal{D}_r^2 + 4A'\mathcal{D}_r + e^{-2A}\square \right] \mathbf{a}^a - \left[V|_c^a - \mathcal{R}_{bcd} \bar{\Phi}'^b \bar{\Phi}'^d + \frac{4(\bar{\Phi}'^a V_c + V^a \bar{\Phi}'_c)}{3A'} + \frac{16V \bar{\Phi}'^a \bar{\Phi}'_c}{9A'^2} \right] \mathbf{a}^c = 0, \quad (43)$$

with boundary conditions (suppressing the index i of $\lambda_{(i)}$)

$$\begin{aligned} & \left[\delta_b^a + e^{2A}\square^{-1} \left(V^a - 4A'\bar{\Phi}'^a - \lambda^a|_c \bar{\Phi}'^c \right) \frac{2\bar{\Phi}'_b}{3A'} \right] \mathcal{D}_r \mathbf{a}^b \Big|_{r_i} = \\ & \left[\lambda^a|_b + \frac{2\bar{\Phi}'^a \bar{\Phi}'_b}{3A'} + e^{2A}\square^{-1} \frac{2}{3A'} \left(V^a - 4A'\bar{\Phi}'^a - \lambda^a|_c \bar{\Phi}'^c \right) \left(\frac{4V \bar{\Phi}'_b}{3A'} + V_b \right) \right] \mathbf{a}^b \Big|_{r_i}. \end{aligned} \quad (44)$$

1. Superpotential formalism

In the special case where there is a superpotential W , we have that Eqs. (43) and (44) become

$$\left[\left(\delta_b^a \mathcal{D}_r + W|_b^a - \frac{W^a W_b}{W} - \frac{8}{3} W \delta_b^a \right) \left(\delta_c^b \mathcal{D}_r - W|_c^b + \frac{W^b W_c}{W} \right) + \delta_c^a e^{-2A} \square \right] \mathbf{a}^c = 0, \quad (45)$$

and

$$\begin{aligned} & \left[\delta^a_b + e^{2A} \square^{-1} \left(\lambda^a_{|c} - W^a_{|c} \right) \frac{W^c W_b}{W} \right] \mathcal{D}_r \mathbf{a}^b \Big|_{r_i} = \\ & \left[\lambda^a_{|b} - \frac{W^a W_b}{W} + e^{2A} \square^{-1} \left(\lambda^a_{|c} - W^a_{|c} \right) \frac{W^c W_d}{W} \left(W^d_{|b} - \frac{W^d W_b}{W} \right) \right] \mathbf{a}^b \Big|_{r_i}, \end{aligned} \quad (46)$$

respectively.

D. Boundary masses

When the superpotential is known, it is convenient to write

$$N^d_b \equiv W^d_{|b} - \frac{W^d W_b}{W}, \quad (47)$$

$$\lambda_{(1)|c}^a \equiv W^a_{|c} \Big|_{r_1} + (m_1^2)^a_c, \quad (48)$$

$$\lambda_{(2)|c}^a \equiv W^a_{|c} \Big|_{r_2} - (m_2^2)^a_c. \quad (49)$$

Notice the different sign in the definition. It is convenient to rewrite the bulk equation for the fluctuations in the compact form

$$\left[e^{-4A} (\delta^a_b \mathcal{D}_r + N^a_b) e^{4A} (\delta^b_c \mathcal{D}_r - N^b_c) + \delta^a_c e^{-2A} \square \right] \mathbf{a}^c = 0, \quad (50)$$

and the boundary conditions as

$$\begin{aligned} & \left[\delta^a_b + e^{2A} \square^{-1} (m_1^2)^a_c \frac{W^c W_b}{W} \right] \mathcal{D}_r \mathbf{a}^b \Big|_{r_1} = \\ & \left[(m_1^2)^a_b + N^a_b + e^{2A} \square^{-1} (m_1^2)^a_c \frac{W^c W_d}{W} N^d_b \right] \mathbf{a}^b \Big|_{r_1}, \end{aligned} \quad (51)$$

and

$$\begin{aligned} & \left[\delta^a_b - e^{2A} \square^{-1} (m_2^2)^a_c \frac{W^c W_b}{W} \right] \mathcal{D}_r \mathbf{a}^b \Big|_{r_2} = \\ & \left[- (m_2^2)^a_b + N^a_b - e^{2A} \square^{-1} (m_2^2)^a_c \frac{W^c W_d}{W} N^d_b \right] \mathbf{a}^b \Big|_{r_2}, \end{aligned} \quad (52)$$

respectively.

The matrices m_i^2 encode the degree of arbitrariness in the definition of the boundary terms. These matrices do not necessarily preserve the same amount of (internal) global symmetries as the bulk sigma-model action. In particular, they provide a source of explicit breaking for possible residual internal symmetries of the sigma-model action. Taking their entries to be large ensures that possible pseudo-Goldstone bosons (arising by the spontaneous breaking of such internal symmetries) are effectively removed from the spectrum. Hence, it is convenient to take the limit in which m_i^2 are diagonal, and their eigenvalues are all positive and divergent, so that the boundary terms reduce to

$$\left[e^{2A} \square^{-1} \frac{W^c W_d}{W} \right] (\delta^d_b \mathcal{D}_r - N^d_b) \mathbf{a}^b \Big|_{r_i} = \delta^c_b \mathbf{a}^b \Big|_{r_i}. \quad (53)$$

One can think of using this procedure also in the case when the physical system does not have explicit physical cutoffs. In this case, this procedure can be thought of as a regulator: one computes the spectrum at finite cutoffs, and then studies how the physical spectrum changes when extrapolating the values of r_i to their natural boundaries (either towards the UV, or towards a singularity/end-of-space boundary in the IR).

IV. APPLICATIONS

In this section, we apply the formalism developed in the previous sections to a few examples, in order to demonstrate how the algorithm that we propose works, as well as to verify that our results agree with the literature when applicable. Although the examples we study here are all simple, in the sense that they consist of only one scalar with a canonical sigma-model metric, the formalism is also applicable to more general cases with several scalars whose sigma-model metric is non-trivial. Indeed, most applications, such as various consistent truncations from supergravity, will fall into this more general category, and it was with this in mind that the formalism of the previous sections was developed.

A. Example A: quadratic superpotential and Goldberger-Wise mechanism

The first example we consider is related to the Goldberger-Wise stabilization mechanism [12], written in the form of [13]. This example has two advantages. It is peculiarly simple, in that there is only one scalar Φ with canonical kinetic term, and has a very simple superpotential. And it has been extensively studied in the literature, thus providing us with a way to check that the formalism used here yields results that are consistent with other approaches. Also, there is a sense in which, in some limit, any physical system near a fixed point resembles it.

The superpotential is

$$W = -\frac{3}{2} - \frac{\Delta}{2}\Phi^2, \quad (54)$$

so that the potential is

$$V = -3 + \frac{1}{2}(\Delta^2 - 4\Delta)\Phi^2 - \frac{1}{3}\Delta^2\Phi^4. \quad (55)$$

Notice the form of the quadratic term in the potential as a function of the parameter Δ , which yields the well-known result that the five-dimensional mass is related to the scaling dimension of the dual operators by $M_5^2 L^2 = \Delta(\Delta - 4)$, and hence provides a natural interpretation for the parameter Δ . The boundary potentials are

$$\lambda_{(1)} = -\frac{3}{2} - \frac{\Delta}{2}\Phi_1^2 - \Delta\Phi_1(\Phi - \Phi_1) - \frac{1}{2}(\Delta - m_1^2)(\Phi - \Phi_1)^2, \quad (56)$$

$$\lambda_{(2)} = -\frac{3}{2} - \frac{\Delta}{2}\Phi_2^2 - \Delta\Phi_2(\Phi - \Phi_2) - \frac{1}{2}(\Delta + m_2^2)(\Phi - \Phi_2)^2. \quad (57)$$

The differential equations and boundary terms for the background are

$$\bar{\Phi}' = -\Delta\bar{\Phi}, \quad (58)$$

$$A' = 1 + \frac{\Delta}{3}\bar{\Phi}^2, \quad (59)$$

$$A' - 1 - \frac{\Delta}{3}\Phi_1^2 \Big|_{r_1} = 0, \quad (60)$$

$$A' - 1 - \frac{\Delta}{3}\Phi_2^2 \Big|_{r_2} = 0, \quad (61)$$

$$\bar{\Phi}' + \Delta\Phi_1 \Big|_{r_1} = 0, \quad (62)$$

$$\bar{\Phi}' + \Delta\Phi_2 \Big|_{r_2} = 0. \quad (63)$$

The solution is hence

$$\bar{\Phi}(r) = \Phi_1 e^{-\Delta(r-r_1)}, \quad (64)$$

$$r_2 - r_1 = -\frac{1}{\Delta} \ln \frac{\Phi_2}{\Phi_1}, \quad (65)$$

$$A = a_0 + r - \frac{1}{6}\Phi_1^2 e^{-2\Delta(r-r_1)}, \quad (66)$$

where a_0 , and Φ_1 are integration constants. As a result, a big hierarchy between the (physical) UV and IR scales can originate from a small value of Δ (which is protected), and with natural choices of the unprotected $\Phi_2/\Phi_1 \sim \mathcal{O}(1)$,

hence removing possible sources of fine-tuning. The constant a_0 can always be reabsorbed into a redefinition of the four-dimensional units, and hence can be chosen to be $a_0 = 0$, so that when $\Phi_{1,2} \rightarrow 0$ one recovers exactly the standard form of the AdS case with unit curvature. It is often convenient to change variable according to $r = -\ln z$, so that with $L_0 < z < L_1$ one finds that

$$\frac{L_0}{L_1} = e^{-r_2+r_1} = \left(\frac{\Phi_2}{\Phi_1}\right)^{\frac{1}{\Delta}}. \quad (67)$$

The value of L_0 and L_1 are the UV and IR (length) scales, and in this form it is manifest that an exponential hierarchy is naturally generated.

1. Spectrum

Before discussing the spectrum, let us make two observations. In the limit in which $\Delta \rightarrow 0$ the fluctuations φ and h decouple from each other. Furthermore, in this limit the field Φ has trivial bulk dynamics, while the background becomes exactly AdS. As a consequence of these observations, the vacuum is determined by two arbitrary, non-dynamical quantities $\bar{\Phi} = \Phi_1 = \Phi_2$ and $r_2 - r_1$. They correspond to two moduli. In this limit, one expects (at least in this semi-classical analysis) the presence of two massless states in the scalar spectrum, associated with these moduli parameterizing the space of (inequivalent) vacua.

The non-vanishing of Δ yields a background which is not AdS, at least in the IR. Effectively, this corresponds to an explicit breaking of scale invariance, and is ultimately responsible for the dynamical stabilization of the finite hierarchy $r_2 - r_1$. As a consequence, one expects the second scalar to stay light even for very large choices of m_i^2 , and its mass to vanish with Δ and Φ_1 . In the language of the AdS/CFT correspondence, $4 - \Delta$ is the dimension of a dual operator with coupling proportional to Φ_1 , the insertion of which breaks explicitly scale invariance.⁵

Because we are interested in the case in which the background is at least approximately AdS, we choose Φ_1 to be small. By inspection of the bulk equation, of the background warp factor A , of the superpotential W , and of its first and second field derivatives W_Φ and $W_{\Phi\Phi}$, it is apparent that one can expand the fluctuations in powers of

$$x \equiv \Delta^2 \Phi_1^2 e^{2\Delta r_1}, \quad (68)$$

by writing

$$\mathfrak{a}(r) = \mathfrak{a}_0(r) + x \mathfrak{a}_1(r) + \dots, \quad (69)$$

and replacing in Eq. (50). At the leading $\mathcal{O}(x^0)$, the bulk equation becomes

$$0 = \left[(\partial_r - \Delta + 4) (\partial_r + \Delta) + e^{-2r} q^2 \right] \mathfrak{a}_0^k(r), \quad (70)$$

where the superscript k refers to the heavy KK-modes, and the boundary conditions

$$0 = - (m_1^2 - \partial_r - \Delta) \mathfrak{a}_0^k - \frac{2}{3} m_1^2 \frac{e^{2r} W_\Phi^2}{q^2} (\partial_r + \Delta) \mathfrak{a}_0^k \Big|_{r_1}, \quad (71)$$

$$0 = - (-m_2^2 - \partial_r - \Delta) \mathfrak{a}_0^k + \frac{2}{3} m_2^2 \frac{e^{2r} W_\Phi^2}{q^2} (\partial_r + \Delta) \mathfrak{a}_0^k \Big|_{r_2}, \quad (72)$$

where we kept explicitly terms proportional to $W_\Phi = -\Delta \Phi_1 e^{-\Delta(r-r_1)}$ for reasons that will become clear soon.

The heavy states can be discussed by looking at the solutions to these equations setting $W_\Phi|_{r_i} = 0$. The solution to the bulk equations is

$$\mathfrak{a}_0^k = e^{-2r} \left[c_1 J_{2-\Delta}(e^{-r} q) + c_2 Y_{2-\Delta}(e^{-r} q) \right], \quad (73)$$

⁵ Notice that the unitarity bounds imply that the five-dimensional mass is $M_5^2 L^2 \geq -4$, which is automatically true provided that the superpotential is given by Eq. (54). This means that when $\Delta < 1$, the only possible interpretation is that the dual operator has dimension $4 - \Delta$.

with c_i determined by the boundary conditions

$$0 = - (m_1^2 - \partial_r - \Delta) \mathfrak{a}_0 \Big|_{r_1}, \quad (74)$$

$$0 = - (-m_2^2 - \partial_r - \Delta) \mathfrak{a}_0 \Big|_{r_2}. \quad (75)$$

The details of the spectrum depend on the specific choice of the m_i^2 terms. Yet, in general the mass gap is related to the zeros of the Bessel functions $J_{2-\Delta}$, at least for $r_2 \gg r_1$. The mass gap is hence $\pi e^{-r_1} = \pi/L_1$, as it is sensible to expect.

Conversely, the lightest states must be treated by keeping the term depending on $1/q^2$ in the boundary conditions, while dropping the one proportional to q^2 in the bulk equations. The reason is that in the limit where $W_\Phi \rightarrow 0$ we expect a massless state to be present. Hence, the lightest state will itself have mass $\mathcal{O}(x)$. It is then convenient to write $q^2 \equiv x\tilde{q}^2$ and $W_\Phi^2 = x e^{-2\Delta r}$. The bulk equation hence becomes, at $\mathcal{O}(x^0)$:

$$0 = \left[(\partial_r - \Delta + 4) (\partial_r + \Delta) \right] \mathfrak{a}_0^d, \quad (76)$$

and the boundary conditions

$$0 = - (m_1^2 - \partial_r - \Delta) \mathfrak{a}_0^d - \frac{2}{3} m_1^2 \frac{e^{2r(1-\Delta)}}{\tilde{q}^2} (\partial_r + \Delta) \mathfrak{a}_0^d \Big|_{r_1}, \quad (77)$$

$$0 = - (-m_2^2 - \partial_r - \Delta) \mathfrak{a}_0^d + \frac{2}{3} m_2^2 \frac{e^{2r(1-\Delta)}}{\tilde{q}^2} (\partial_r + \Delta) \mathfrak{a}_0^d \Big|_{r_2}, \quad (78)$$

where the superscript \mathfrak{a}_0^d indicates that we interpret this state as a light (pseudo-)dilaton. The bulk equation is satisfied by

$$\mathfrak{a}_0^d = c_1 e^{-\Delta r} + c_2 e^{-(4-\Delta)r}. \quad (79)$$

Notice that, because of the presence of q^2 in a denominator in the boundary conditions, it is sufficient to solve the leading-order bulk equation in order to derive the (subleading) $\mathcal{O}(x)$ value for the mass of the dilaton.

One has to solve two algebraic equations to determine the ratio c_1/c_2 and \tilde{q} . We do so for the extreme case $m_i^2 \rightarrow \infty$. The result is

$$\tilde{q}^2 = \frac{4e^{-2(r_1+r_2)} (e^{2r_1} - e^{2r_2}) (\Delta - 2)}{3 (e^{2(\Delta-2)r_1} - e^{2(\Delta-2)r_2})}, \quad (80)$$

which for $r_1 = 0$ yields the dilaton mass

$$m_d^2 = 4\Delta^2 \Phi_1^2 \frac{2-\Delta}{3} \frac{1 - e^{-2r_2}}{1 - e^{2(\Delta-2)r_2}}. \quad (81)$$

This result, obtained so easily, is in splendid agreement with the literature [23, 24].

Notice that in deriving the mass of the light (pseudo-)dilaton we did not make any assumptions about Δ , aside from requiring it to be positive. Hence all of the above holds for generic Δ , not just for the $\Delta \ll 1$ case. Also, the last factor in Eq. (81), dependent on r_2 , assumes unit value for large $r_2 \gg 0$, provided $\Delta < 2$. It turns negative when $\Delta > 2$, at which point however the negative sign is compensated by the $\Delta - 2$ factor, ensuring that the mass is positive. In this case the mass vanishes for asymptotically large values of r_2 .

2. Zero modes

We devote this short subsection to analyzing more in detail the limits in which $m_d^2 \rightarrow 0$. For $m_i^2 = 0$, the boundary conditions in Eq. (77) and Eq. (78) reduce to

$$(\partial_r + \Delta) \mathfrak{a}_0^d \Big|_{r_i} = 0, \quad (82)$$

and there is a massless state with bulk profile $\mathfrak{a}_0^d \propto e^{-\Delta r} = z^\Delta$. In the $\Delta \rightarrow 0$ case this profile becomes constant.

The existence of this zero-mode for $m_i^2 \rightarrow 0$ is a very general property for any system of n scalars. If a superpotential description exists, and we apply the formalism of Eq. (50), (51) and (52), notice that

$$\left(\delta^a{}_c \mathcal{D}_r - N^a{}_c \right) \frac{W^c}{W} = 0. \quad (83)$$

This implies that $\tilde{\mathbf{a}}^a = W^a/W$ always solves Eq. (50) for $q^2 = 0$.

$\tilde{\mathbf{a}}^a$ satisfies the boundary conditions obtained by setting $m_i^2 = 0$, which reduce precisely to

$$\left(\delta^a{}_c \mathcal{D}_r - N^a{}_c \right) \tilde{\mathbf{a}}^c \Big|_{r_i} = 0. \quad (84)$$

This observation shows explicitly that there is always a massless state when $m_i^2 = 0$, which is the one discussed at the beginning of this subsection. Notice however that this is in general the result of fine-tuning of m_i^2 , and caution must be used in interpreting this result.

In the other extreme, more physical case, in which $m_i^2 \rightarrow +\infty$, the boundary conditions in Eq. (77) and Eq. (78) become (for $x \ll 1$)

$$\frac{2e^{2r(1-\Delta)}}{3\tilde{q}^2} (\partial_r + \Delta) \mathbf{a}_0^d + \mathbf{a}_0^d \Big|_{r_i} = 0. \quad (85)$$

Setting $r_1 = 0$ for simplicity, the solution is

$$\mathbf{a}_0^d \propto e^{(\Delta-4)r} - \frac{e^{-\Delta r} (1 - e^{2(\Delta-1)r_2})}{1 - e^{2r_2}}, \quad (86)$$

which reduces to $\mathbf{a}_0^d \propto e^{-4r} + e^{-2r_2}$ in the $\Delta \rightarrow 0$ case (in which this is a massless state).

Notice a very interesting fact: taking the limit $r_2 \rightarrow +\infty$ in Eq. (86) automatically yields a profile that corresponds to keeping only the subleading behavior in the generic solution. This shows for a concrete example that the procedure we are implementing automatically reproduces the results obtained with the more widely adopted idea of defining the spectrum only in the absence of a UV boundary, by imposing that the solutions to the fluctuation equations vanish at infinity as fast as possible with r . Finally notice one important fact about Eq. (86). The bulk profile of the massless state that is present when $\Delta \rightarrow 0$ is correctly identified by first studying the $\Delta \neq 0$ limit, in which case the corresponding state is light but not massless, and then taking the $\Delta \rightarrow 0$ limit at the end of the calculation.

3. Discussion

Let us discuss now what happens for generic values of m_i^2 . We keep working under the assumption that $x \ll 1$, so that Eqs. (70), (71) and (72), which describe the generic bulk profiles, still hold. Let us consider first the lightest state. Solving for generic values of m_i^2 , and for simplicity setting $r_1 = 0$, yields

$$m_d^2 = \frac{4\Delta^2 \Phi_1^2 (1 - e^{-2r_2}) (\Delta - 2) m_1^2 m_2^2}{3 (e^{2(\Delta-2)r_2} m_1^2 (m_2^2 + 2\Delta - 4) - (m_1^2 - 2\Delta + 4) m_2^2)} \quad (87)$$

This result is completely general (i.e. valid for any m_i^2 and Δ). Notice that, as we already know, for $m_i^2 = 0$ one finds a massless state, and for $m_i \rightarrow +\infty$ the finite result in Eq. (81), suppressed by x . The former choice should be avoided, if what one is trying to understand is whether the scenario in question predicts the existence of a light scalar, irrespectively of the details of the UV dynamics and of the coupling to other sectors of the complete theory of interest (such as the SM).

Also, a pathology appears when taking $|m_i^2| = 4 - 2\Delta$. The reason for this lies in the way in which we wrote the boundary terms. Consider for example the UV term: looking at the coefficient of the $(\Phi - \Phi_1)^2$ term, replacing such pathological choice one has $m_1^2 - \Delta = -4 + \Delta$. In this case, what is happening is that this is the choice that would render massless the excitations around the other solution to the (second-order) bulk equations for the background, which has scaling dimension $4 - \Delta$ rather than Δ . Therefore, this choice should also be avoided.

One special comment about unitarity. The way in which we wrote the system, in terms of a quadratic superpotential, yields a potential in which the (five-dimensional) mass term $M_5^2 = \Delta^2 - 4\Delta > -4$ is always above the unitarity bounds, irrespectively of the value of Δ . The bound is saturated for $M_5^2 = -4$, or $\Delta = 2$, in which case the theory is close to a special transition point that we will discuss at length elsewhere. In proximity of this point, the mass is anomalously

suppressed. Yet, we did add boundary terms with arbitrary parameters m_i^2 , which distort the spectrum, and one has to check that in doing so no tachyon state has been added. One can easily verify that for $r_2 \rightarrow +\infty$, and $\Delta < 2$ this implies that one must enforce the choice $m_1^2 \geq 0$. For $r_2 \rightarrow +\infty$, and $\Delta > 2$ one conversely must impose $m_2^2 \geq 0$. In general, given a choice of Δ and m_i^2 one has to verify that no tachyon is present.

Finally, we turn to the heavy modes, which have bulk profile \mathfrak{a}_0^k up to $\mathcal{O}(x)$ corrections. We already stated (in Section IV A 1) that the heavy modes form towers with separation πe^{r_1} . One has to clarify where the second light state mentioned several times ends up, for general values of m_i^2 . The general solution can be obtained by simply solving for the integration constants and for q^2 in the equations for \mathfrak{a}_0^k . This is a somewhat intricate exercise, which is not very illuminating. Yet, there exists an interesting limiting case: for $m_1^2 \rightarrow +\infty$ and $m_2^2 \rightarrow 0$ one finds that one light state has the mass m_d^2 discussed earlier on in (81), while at the same time an exactly massless state also is present. Hence, in this case it is clear that two light scalars are present, while the heavy states start appearing with masses proportional to $\mathcal{O}(\pi e^{-r_1})$. Effectively, what is happening is that the whole tower is shifted down in this limit, and the first excited state in the tower becomes parametrically light. The numerical study performed in the next section will make these observations more clear.

B. Example B: a consistent truncation of type IIB supergravity

Let us now consider a different example. We still consider the case where only one scalar Φ is present, and the sigma-model is trivial. But now the superpotential is

$$W = -\frac{3}{4} \left(1 + \cosh 2\sqrt{\frac{\Delta}{3}}\Phi \right), \quad (88)$$

so that the potential is

$$V = -3 \cosh \left[\sqrt{\frac{\Delta}{3}}\Phi \right]^4 + \frac{3}{8} \Delta \sinh \left[2\sqrt{\frac{\Delta}{3}}\Phi \right]^2 \quad (89)$$

$$\simeq -3 + \left(-2\Delta + \frac{\Delta^2}{2} \right) \Phi^2 + \dots, \quad (90)$$

where in the last expression we expanded for small Φ . Notice how this expansion is in agreement with the potential of the previous sections, at leading order. The difference is important only away from the AdS fixed point $\Phi = 0$.

The solution to the bulk equation is

$$\bar{\Phi} = \sqrt{\frac{3}{\Delta}} \operatorname{arctanh} e^{-\Delta r + c_1}, \quad (91)$$

and we can always choose $c_1 = 0$ for simplicity, setting the radial coordinate in such a way that $\bar{\Phi}$ diverges for $r \rightarrow 0$. The warp factor is

$$A = \frac{1}{2\Delta} \ln(-1 + e^{2\Delta r}). \quad (92)$$

Replacing $r = -\log z$, and expanding for small $z \rightarrow 0$,

$$\bar{\Phi} = \sqrt{\frac{3}{\Delta}} z^\Delta, \quad (93)$$

not surprisingly. Notice however that, as opposed to the what we did in the previous sections, the coefficient in front of z^Δ is now fixed. There is no free parameter analogous to Φ_1 . The integration constant that we set to zero would change this coefficient, but it would also change the position of the singularity in the IR. In this sense, if we think of $\bar{\Phi} \neq 0$ in terms of spontaneous symmetry-breaking, in this model there is a direct link between the formation of such a symmetry-breaking condensate and the end of space in the IR (which one would like to associate with confinement).

For $\Delta = 3$ the potential becomes

$$V = \frac{3}{4} \cosh^2 \Phi (-5 + \cosh 2\Phi), \quad (94)$$

which is the five-dimensional potential obtained by consistently truncating type IIB supergravity on a Sasaki-Einstein manifold discussed in [30] in the context of holographic superconductivity, and that has a very long history in the context of truncations of type IIB to 5D supergravity intended to yield the duals of controlled deformations of $\mathcal{N} = 4$ SYM, being a special case of the GPPZ flows [31]. In this case the full lift to 10-dimensional type-IIB supergravity is known. Specifying $\Delta = 3$, the background scalar is

$$\bar{\Phi} = \operatorname{arctanh} e^{-3r}, \quad (95)$$

and the five-dimensional warp factor is

$$A = \frac{1}{6} \log(-1 + e^{6r}), \quad (96)$$

where we chose an integration constant in such a way that $A \rightarrow r$ far in the UV. Notice how this background is practically the same as the one discussed in the previous example, with the choices $\Delta = 3$, $r_1 = 0$ and $\Phi_1 = 1$, aside from a very narrow region near the IR boundary, where a singular behavior appears.

For our purposes, it is interesting to study the spectrum with the background solution given by the first-order equations, mainly because of its simplicity, which will help us elucidate on the effect of performing the calculation with the explicit boundary terms in presence of singular backgrounds. Unfortunately, because of the singularity in the IR, there is no small parameter controlling the VEV to expand in. As a consequence, one must rely on numerically solving the fluctuation equations. Yet, we can obtain some semi-quantitative information from our previous results. We will perform the numerical calculations at finite $r_2 \gg r_1 > 0$, and then extrapolate to the cases where the IR and UV cutoffs are removed ($r_1 \rightarrow 0$ and $r_2 \rightarrow +\infty$).

As long as we choose $r_1 \gg 0$, the background being hardly different from the previous case, we expect the same results to hold. In particular, because the non-trivial departure is localized very close to the IR boundary, we do not expect any interesting changes in the spectrum of the heavy modes, for which all the approximations we made should still hold, up to a possible overall shift of the spectrum. However, more interesting is the case of the light state. In this case, the fact that the VEV of Φ diverges near the IR boundary means that the approximations we made might not hold. It is hence very interesting to see what happens when $r_1 \rightarrow 0$. We will study this numerically in the next section.

We conclude with a few comments on the (real) supergravity background generated by this action. It must be noted that this system has been extensively studied in the literature, and yields a background that is well-known to be a badly singular limit of the GPPZ system, which fails to satisfy even the modest demands of [32].

One way of seeing explicitly that a problem is present is the following. The lift to 10 dimensions yields the metric [30]⁶

$$ds_{10}^2 = \cosh \Phi (e^{2A} dx_{1,3}^2 + dr^2) + d\Omega_5, \quad (97)$$

with $d\Omega_5$ the internal metric, which depends in general on Φ and r , besides the coordinates of the five-dimensional internal manifold.

The internal-space structure of the metric is not very important for the present discussion, what matters is that we know the warp factor $\cosh \Phi$ needed to lift the five-dimensional metric to ten dimensions. This information allows to use the background and compute the Wilson loop. One must solve the classical problem of determining the configuration of a (probe) string, the end-points of which are bounded to a D3-brane fixed at some radial position, that we can identify with the UV cutoff r_2 [34], by minimizing the Nambu-Goto action.

Following the standard procedure (see also [21]), one first defines the functions $f^2 = g_{tt}g_{xx}$ and $g^2 = g_{tt}g_{rr}$, in terms of the elements of the 10-dimensional metric. The separation between the end-points of the string L_{QQ} can be computed, by using the auxiliary effective potential $V_{eff}^2(r) = f^2(r)(f^2(r) - f^2(r_0))/(g^2(r)f^2(r_0))$, as a function of the minimum value of the radial direction r_0 reached by the string in its fall into the radial direction, via

$$L_{QQ} = 2 \int_{r_0}^{r_2} \frac{d\rho}{V_{eff}(\rho)}. \quad (98)$$

We are not going to do this exercise here, but we want to observe the fact that in the case we are discussing one finds that

$$f^2 = (1 - e^{-6r})^{-1}(-1 + e^{6r})^{2/3} \quad (99)$$

⁶ The dilaton is constant hence there is no real difference between string frame and Einstein frame.

is not monotonic, but rather has a minimum at $\bar{r} = (1/6) \ln(3/2)$. (Also, f^2 diverges at the singularity, which means that the model fails to satisfy any of the criteria in [32], as anticipated). In turns, this means that the string cannot fall all the way into the bulk towards the singularity, but can at most reach down to \bar{r} . Ultimately, this means that the singularity is bad enough that probing the system with extended objects is going to yield unphysical results, and hence one should not think of this as a complete model, in which the background captures all the physics of the dual confining theory. A resolution of the IR singularity would be needed.

1. The $\Delta = 1$ case

The spectrum of the actual GPPZ system for less pathological cases has been discussed for instance in [33], and the spectrum of this model for $\Delta = 1$ is discussed for instance in the first reference in [25]. We briefly digress here and redo this last calculation, which can be performed analytically, and which is of marginal relevance to the rest of the paper. We apply our procedure, keeping $r_2 \gg r_1$ fixed, and considering values of $r_1 \ll 1$, very close to the singularity. We limit ourselves to the $m_i \rightarrow +\infty$ case. By using the fact that $N \equiv W_{\Phi\Phi} - (W_\Phi)^2/W = -\Delta$, and specifying $\Delta = 1$, the bulk equation for the fluctuations is

$$(-1 + e^{2r}) \mathbf{a}''(r) + 4e^{2r} \mathbf{a}'(r) + (q^2 + 1 + 3e^{2r}) \mathbf{a}(r) = 0, \quad (100)$$

subject to the boundary conditions

$$-\frac{2}{q^2} (\partial_r + 1) \mathbf{a} - \mathbf{a} \Big|_{r_i} = 0. \quad (101)$$

By solving this equation and imposing the boundary conditions, and then taking the $r_2 \rightarrow +\infty$ limit, and $r_1 \rightarrow 0$, one finds that the spectrum is given by

$$m_n^2 = 4n(n+1), \quad \text{for } n = 1, \dots, \infty, \quad (102)$$

in agreement with the literature.⁷ Notice in particular that there are no parametrically light states: the lightest state has a mass $m^2 = 8$, while extrapolating (outside its regime of validity) Eq. (81) computed in Example A, with $\Delta = 1$ and $\Phi_1 = \sqrt{3}$ would yield $m_d^2 = 4$.

Concluding this short exercise, let us make two comments inspired by the fact that these results agree with the literature. First of all, the boundary conditions we use do not rely on the concepts of *normalizability* and/or *regularity*. They are simply defined algebraically, in terms of the background functions, and there is no need to analyze on a model-by-model basis the fluctuations near special points, in order to decide what is physically acceptable and what not: the whole procedure is automatically taking care of this, because the boundary actions implement (both on the background and on the fluctuations) all the physical requirements. Second, for this particular example, the procedure we follow, in which two cutoffs are explicitly present, ultimately yields the same physical results as other procedures, once the limit of removing the cutoffs is taken (provided we take $m_i^2 \rightarrow \infty$). This suggests a general procedure for how to calculate the spectrum in the case of when a singularity may be present for the background, assuming the IR and UV behaviors of the background are not too pathological, and the limit of removing the cutoffs can be taken without difficulties.

C. Example C: a phenomenological model with cubic superpotential

Consider now the following superpotential, for one scalar field Φ with trivial sigma-model:

$$W = -\frac{3}{2} - \frac{\Delta}{2} \Phi^2 + \frac{\Delta}{3\Phi_I} \Phi^3. \quad (103)$$

This amounts to including a cubic correction to the quadratic potential we studied earlier. This superpotential admits two fixed points for $\Phi = \Phi_U = 0$ and $\Phi = \Phi_I$.

⁷ Note, however, that for generic values of m_i^2 , the spectrum would in general be different.

The potential, when expanded near the two fixed points, yields respectively

$$V_U = -3 + \frac{1}{2}\Delta(\Delta - 4)\Phi^2 + \dots, \quad (104)$$

$$V_I = -3 - \frac{2\Delta\Phi_I^2}{3} - \frac{\Delta^2\Phi_I^4}{27} + \frac{1}{2}\Delta\left(\Delta + 4 + \frac{4\Delta\Phi_I^2}{9}\right)(\Phi - \Phi_I)^2 + \dots. \quad (105)$$

In order for the Φ_I to be an attractive IR fixed point, the cosmological constant must be negative and have larger absolute value than it has at the UV fixed point. Which is true for $\Delta\Phi_I^2 > -18$, and in particular for any positive value of Δ .

The solution of the background equations can be obtained by simply integrating

$$\partial_r \Phi = \frac{dW}{d\Phi} = -\frac{\Delta}{\Phi_I}\Phi(\Phi_I - \Phi). \quad (106)$$

We conventionally choose $\Phi_I > 0$. Then the solution for $\Phi > \Phi_I$ or $\Phi < 0$ is

$$\Phi_Q = \frac{\Phi_I}{e^{\Delta(r-r_0)} - 1}, \quad (107)$$

where r_0 is an integration constant. For $r > r_0$ the result is a flow away from the UV fixed point $\Phi = 0$, towards a singularity at $r \rightarrow r_0$ at which $\Phi \rightarrow -\infty$. For $r < r_0$, Φ_Q describes a flow from asymptotically large values of $\Phi > \Phi_I$ near $r \rightarrow r_0$, to the IR fixed point Φ_I when $r \rightarrow -\infty$.

More interesting is the solution obtained when setting the boundary condition so that $0 < \Phi < \Phi_I$. In this case

$$\bar{\Phi} = \frac{\Phi_I}{e^{\Delta(r-r_*)} + 1}, \quad (108)$$

where again r_* is an integration constant, which in this case has a very different meaning. The flow described from $\bar{\Phi}$ connects the two fixed points, running from $\Phi \rightarrow 0$ when $r \gg r_*$, to $\Phi \rightarrow \Phi_I$ when $r \ll r_*$. One can easily see that $\Phi(r_*) = \Phi_I/2$, so that r_* is the scale that separates the regimes in which the theory can be described as a deformation of the two fixed points, respectively.

We are interested in studying the spectrum of the theory defined by the background $\bar{\Phi}$. Strictly speaking, this is continuous. We perform the study by assuming that there exist two hard-wall cutoffs in the IR and UV, such that $r_I \ll r_* \ll r_U$. The scales $r_{I,U}$ are clearly spurious, representing cutoffs put in by hand. In particular, the IR scale should appear as a consequence of a relevant deformation driving the flow away from the IR fixed-point. This requires extending the system, with more scalars being included. Also, the ultimate end-of-space (singularity) will determine the spectrum. In particular, it may be that the physics near the singularity, responsible for its resolution, can be described only by embedding the model into a full 10-dimensional supergravity, or even superstring theory. This is well beyond the scopes of this simple phenomenological model, and hence one should not be too much concerned about the dependence of the masses (in particular of the lightest modes) on r_I . Yet, it is of interest to perform this calculation in order to understand how the spectrum depends on r_* , for fixed choices of $r_{I,U}$. This will offer some guidance as to what happens in actual string-theory models describing the field theory RG flow between UV and IR fixed points.

V. NUMERICAL STUDIES

In this section we present a set of numerical studies of the backgrounds introduced in the previous section. Besides allowing us to check explicitly some of the results, this also allows us to understand how good our approximations are.

A. Quadratic superpotential

We start from the system with quadratic superpotential, considering generic values of Δ . The first thing we want to do is to understand how precise our results in Eqs. (81) and (87) are. Figure 1 shows the dependence of the mass of the dilaton on the dimension Δ , computed in the limit $m_i^2 \rightarrow +\infty$. For (not necessarily very) small values of Φ_1 , the

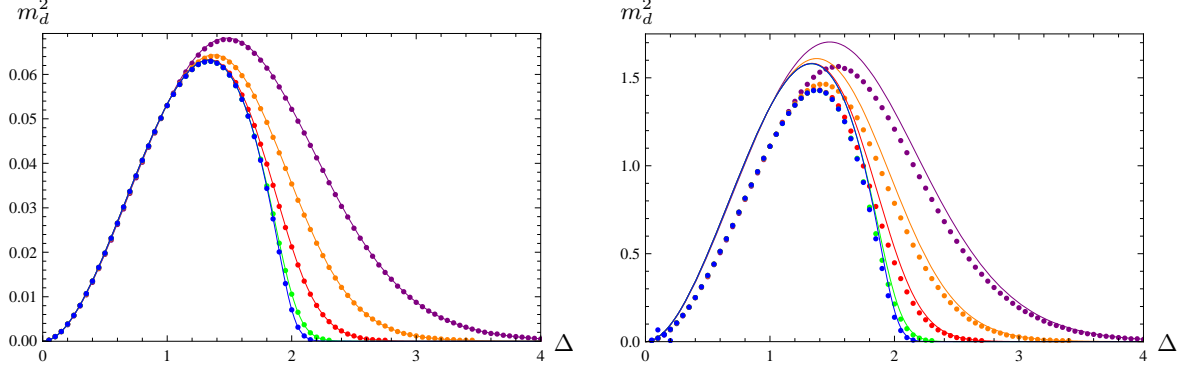


FIG. 1: Numerical results. Mass m_d^2 of the lightest scalar fluctuation as a function of Δ . The localized mass terms are divergent ($m_i^2 \rightarrow +\infty$). All plots with $r_1 = 0$, and $r_2 = 2, 3, 5, 10, 15$ (with faster fall-off for $\Delta > 2$ the higher the UV cutoff). Continuous lines represent the approximation in the body of the paper, while the points are numerical results. The background is chosen to have $\Phi_1 = 0.2$ in the left panel, and $\Phi_1 = 1$ in the right panel.

agreement of the numerical results with Eq. (81) is very remarkable (see the left panel in Figure 1). The agreement deteriorates for larger values of Φ_1 , yet the qualitative features are preserved.⁸

A set of remarkable physics lessons that can be read directly off Eq. (81) are confirmed.

- There are no tachyons. At least at this level, there is no reason to question the stability of the backgrounds we are studying.
- There is unmistakable evidence that two very different behaviors appear for $\Delta > 2$ and $\Delta < 2$, indicating the fact that $\Delta = 2$ is a very special point of the parameter space, with peculiar physical features.
- When $\Delta < 2$, the mass of the light state depends crucially on the dimensionality of the dual operator Δ , on the normalization of the five-dimensional VEV Φ_1 and on the IR scale r_1 , but not on r_2 , the UV cutoff. With the specific choices we made in the plots, it is clear that even at moderate values of r_2 this dependence amounts to a subleading effect.
- The mass m_d^2 is anomalously light not only for $\Delta \ll 1$, which is a well-known and studied result [23, 24], but also when $\Delta \simeq 2$. This is due to the fact that when the limit $\Delta \rightarrow 2$ is taken in Eq. (81), the result $m_d^2 \sim \frac{8\Phi_1^2}{3r_2}$ is suppressed by $1/r_2$.
- For $\Delta > 2$ the mass is very strongly dependent on r_2 , being exponentially suppressed in the limit of $r_2 \rightarrow \infty$ (taking the limit by holding Φ_1 fixed) with $m_d^2 \sim \frac{4}{3}e^{-2(\Delta-2)r_2}(\Delta-2)\Delta^2\Phi_1^2$. Thus, provided r_2 is very large, m_d^2 vanishes, for all practical purposes, for all $\Delta > 2$. Yet, this statement is very cutoff dependent, and needs to be taken with caution.

The more general expression in Eq. (87) taught us a few important subtleties related to these kinds of systems. In particular, we already explained that the five-dimensional sigma-model formalism may yield spectra containing many light states, which have nothing to do with scale invariance. They might be related to the light techni-pions and/or techni-axions of a generic technicolor model, rather than having to do with the techni-dilaton. Of course, because all the global symmetries (including scale invariance of the dual theory) are only approximate, the spectrum results from non-trivial mixing among all possible scalar bound states. This also implies that one has to be very careful in identifying the nature and couplings of the physical (mass eigenstate) states. It is hence a useful exercise to study this problem within this very simplified model, by studying explicitly how the boundary terms m_i^2 distort the scalar spectrum.

This is done in the four panels of Figures 2 and 3. Figure 2 focuses on values of $\Delta = 1$ and $\Phi_1 = 0.2, 1, 3, 5$, for $r_1 = 0$ and $r_2 = 5$. We keep $m_1^2 \rightarrow +\infty$, but vary m_2 . A few very interesting results emerge, which are very general.

⁸ In the right panel of Figure 1, a couple of extra points can be seen for small Δ . Let us note that these are spurious states, in the sense that if one were to take the limit of $r_2 \rightarrow \infty$, their masses would diverge and they would decouple from the rest of the spectrum.

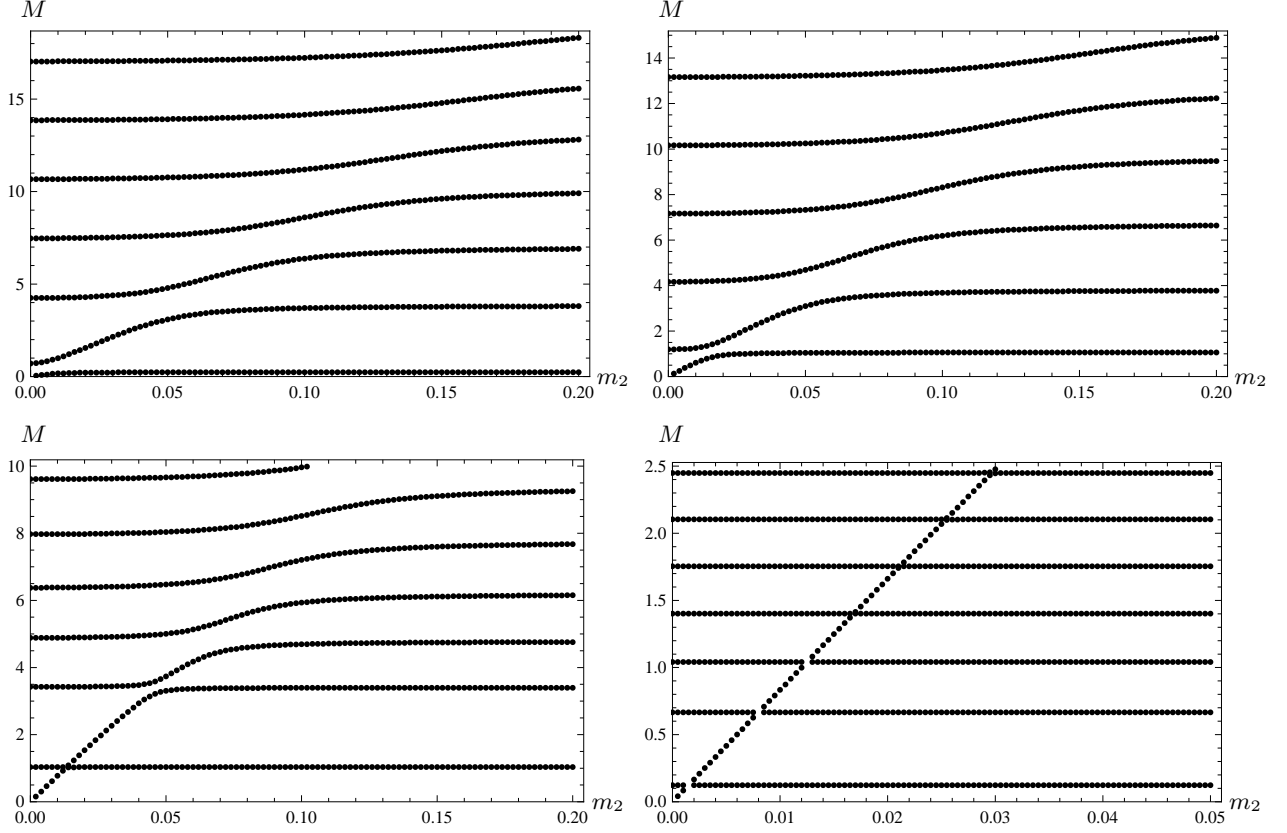


FIG. 2: Numerical results. Mass M of the lightest few scalar states, for $\Delta = 1$, $r_1 = 0$, $r_2 = 5$, plotted as a function of the boundary mass m_2 , for $m_1 \rightarrow +\infty$. The four plots differ for the choice of $\Phi_1 = 0.2, 1, 3, 5$ (left to right and top to bottom).

- When $m_2 \rightarrow 0$, one of the masses vanishes. We already explained the reason for this in Section IV A.
- An interesting level-crossing pattern develops at intermediate values of m_2 . In particular, this shows explicitly how the mixing between the states is very non-trivial. The composition (in terms of the original fluctuations of scalars and metric) of the states corresponding to the pseudo-dilaton is in general very complicated.
- The plots are restricted to the physically acceptable region of parameter space in which $m_2^2 > 0$. However, notice how the mass of the lightest state vanishes as a function of m_2 for $m_2 \rightarrow 0$. If one were to look at large negative values of m_2^2 , the spectrum would contain a tachyon. This means that in setting up one of these models, some attention has to be given not only to how the (super)potential is chosen, to what background solutions one studies, but also to which boundary terms are present.
- It is only for small values of m_2^2 that the spectrum differs significantly from that of the $m_i^2 \rightarrow \infty$ limit. Already for $m_2^2 \sim \mathcal{O}(1)$, the spectrum starts to look the same as for $m_i^2 \rightarrow \infty$, which therefore is a limit that captures the generic behavior. Conversely, taking m_i^2 to be small should be thought of as a kind of fine-tuning.

Figure 3 shows another remarkable fact. By varying Δ , one can see that for small choices of Δ and of m_2 , there are actually two abnormally light states in the spectrum. By looking at the superpotential, it is immediately evident that what is happening in this case is that the two are related to the dilaton and a pseudo-Goldstone boson, the latter emerging from the near flatness of the sigma-model scalar potential. As soon as m_2 and Δ become generic $\mathcal{O}(1)$ numbers, both these states will in general be heavy, with masses dictated by the general IR scale that controls all of the spectrum. However, when $\Delta > 2$ there is always one light state (whose mass is suppressed by the UV cutoff) irrespective of m_2 .

Let us make a final remark for the reader, who might find it very bizarre that the typical scale of the KK masses in the panels in Figures 2 and 3 are so different. We said earlier on that the scale controlling the gaps is simply related to r_1 , and hence one might have expected the heavy states to have very similar masses, up to an overall shift

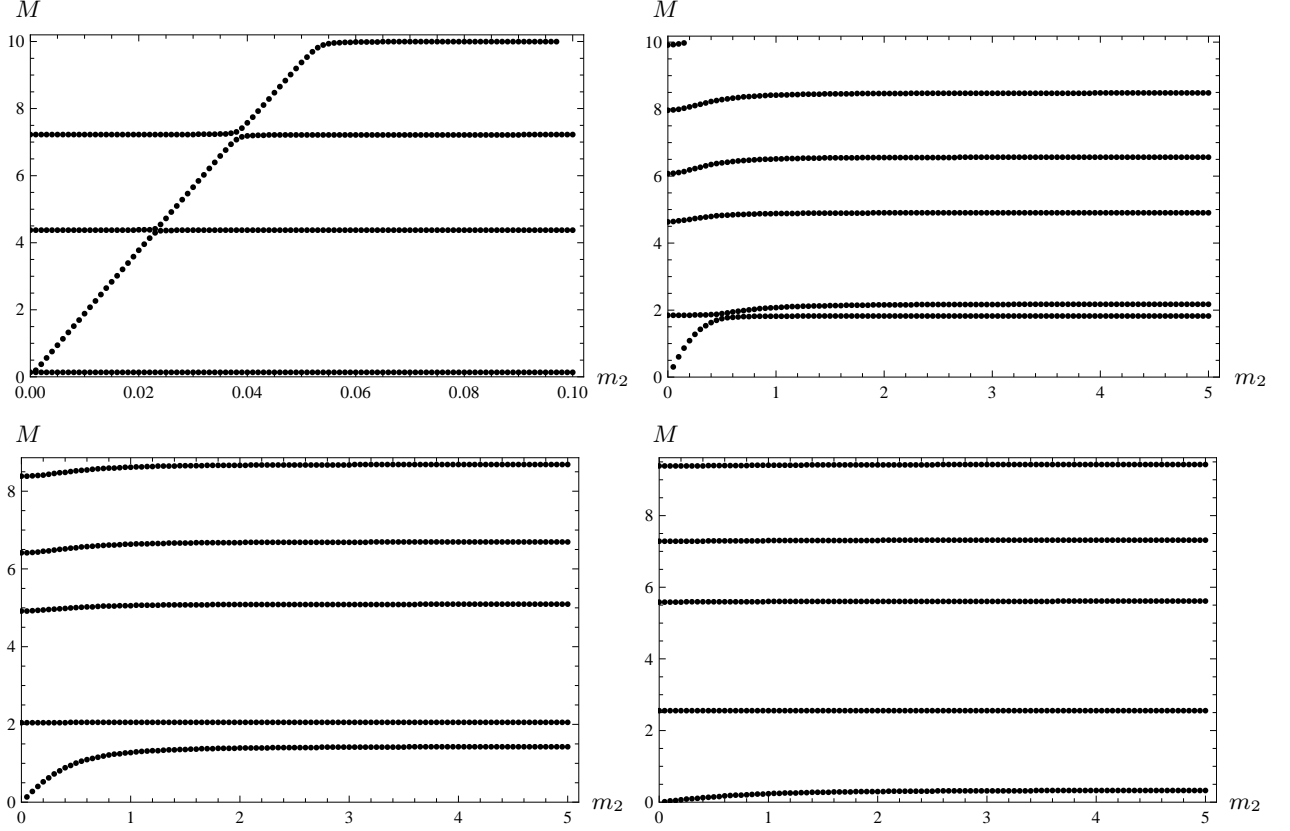


FIG. 3: Numerical results. Mass M of the lightest few scalar states, for $\Phi_1 = 3$, $r_1 = 0$ and $r_2 = 5$, plotted as a function of the boundary mass m_2 , for $m_1 \rightarrow +\infty$. The four plots differ for the choice of $\Delta = 0.2, 1.8, 2, 2.5$ (left to right and top to bottom).

controlled by Φ_1 . While this is not what the figures seem to show, there is indeed no contradiction, for a subtle reason, which ultimately has to do with how we decided to perform the numerical comparison between the three different backgrounds, rather than with the physics. The subtlety emerges from the following observation. In all the backgrounds we are looking at, at asymptotically high values of r the background is characterized by a vanishing small value for $\bar{\Phi}(r)$, and consequently the warp factor $\bar{A}(r) \simeq r + a_0$. We made the choice of setting $a_0 = 0$ in all cases. This ensures that the backgrounds are asymptotically all the same, with the same normalizations for the 4-dimensional Minkowski space-time variables. In this way, the value of r_2 always corresponds to the same UV cutoff scale. However, in the IR the metric is going to change, because $\bar{\Phi}(r) \neq 0$. The larger the value of Φ_1 , the larger the departure from AdS. And, more importantly, the departure will appear at higher values of r . As a consequence, it is not true that by keeping the same value of r_1 in backgrounds with different Φ_1 one is introducing an IR cutoff at the same scale. In other words, the numerical value of r_1 is not an actual physical scale. It can be converted into such a scale only as a function of all the other parameters in the background. We will see a much cleaner example of this later on in Example C, when talking about backgrounds that describe the flow between two fixed points, for which the geometry interpolates between two AdS spaces with different curvature.

B. Example from consistent truncation

Here we will study Example B numerically, and focus in particular on the case when the parameter Δ in the superpotential is fixed to be $\Delta = 3$. As explained before, the model can then be thought of as having a stringy origin. The background develops a singularity in the IR at $r = 0$. However, as shown in Figure 4, the background fields $\bar{\Phi}$ and A are almost indistinguishable from those used in Example A, with the choices $\Delta = 3$ and $\Phi_1 = e^{-\Delta r_1}$, with the exception of a very narrow region close to the singularity.

We study how the spectrum depends on where we put the IR cutoff. In particular we focus on what happens in the limit of letting $r_1 \rightarrow 0$, keeping the UV cutoff fixed. The results are plotted in Figure 5, where we also show the

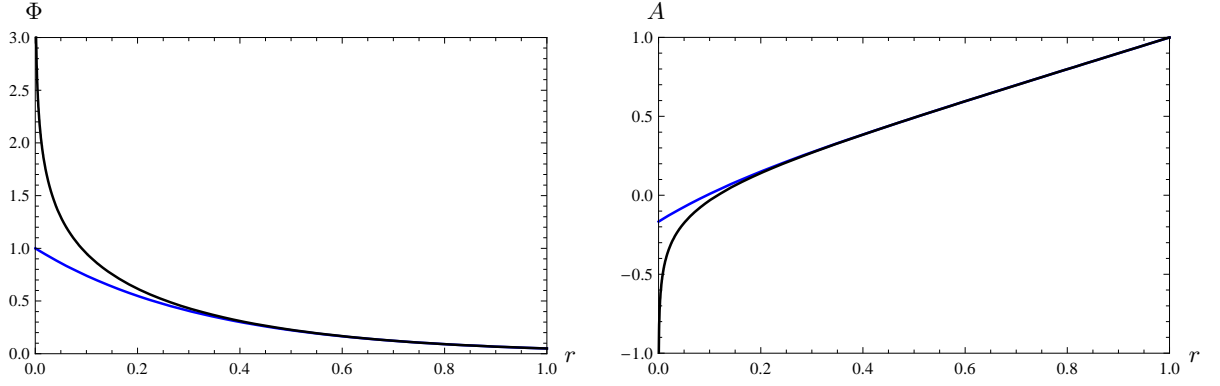


FIG. 4: The background functions Φ (left panel) and A (right panel), used in Example B (black lines), compared to the case discussed in Example A, for the choices $\Delta = 3$ and $\Phi_1 = e^{-\Delta r_1}$ (blue lines).

behavior of the Example A for comparison. There is a tower of KK modes, and a light scalar. Remarkably, despite the presence of an IR singularity, all states in the spectrum converge to their respective (finite) values as the IR cutoff is taken close to the position of the singularity. Also, while a shift in the masses of the heavy states is clearly visible, this effect is very suppressed for the lightest mass. This is a very interesting result, since naively one would expect that the lightest state would be the most sensitive to the IR singularity!

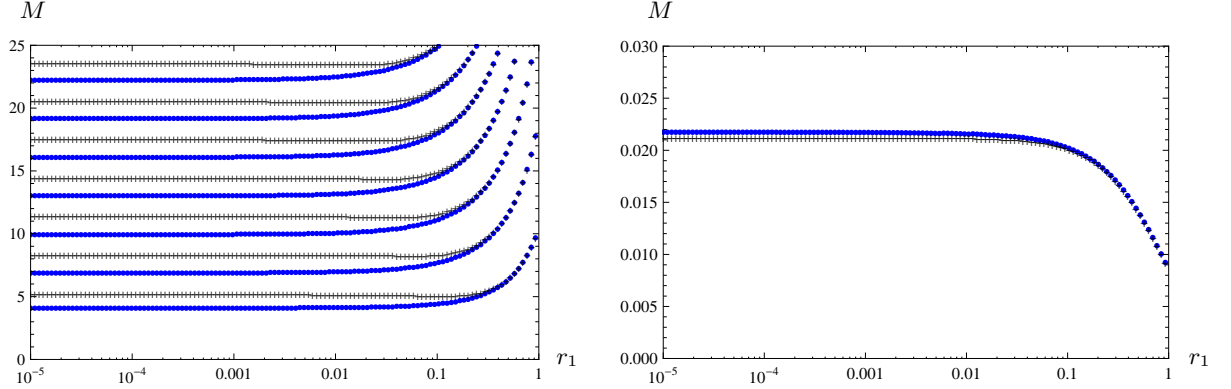


FIG. 5: Numerical results. Mass M of the lightest scalar fluctuations, keeping the UV cutoff fixed at $r_2 = 5$ while varying the IR cutoff r_1 . The right panel shows a detail of the left one, in which the very lightest state is not visible (notice the scale). The plots show the comparison between the results of Example B (black +) and of Example A with $\Delta = 3$ and $\Phi_1 = e^{-\Delta r_1}$ (blue).

In Figure 5, the dependence of the spectrum on the UV cutoff is shown. As can be seen, the mass of the light scalar tends to zero in the limit of infinite r_2 , whereas the rest of the spectrum stabilizes.

C. Phenomenological example: cubic superpotential

This is to be understood as a toy model describing the RG flow between two fixed points. In Figure 7 a set of possible backgrounds for this model are shown, with the background functions $\bar{\Phi}$ and A explicitly plotted as a function of r . The various backgrounds share the same choices of $\Delta = 3$, $\Phi_I = 1$, $r_1 = 0$ and $r_2 = 5$. The different curves are obtained by varying r_* , the value of the radial direction at which the nature of $\bar{\Phi}(r)$ changes. For higher values of Δ , the kink becomes more localized.

Notice that for $r_* \rightarrow +\infty$, the background becomes purely AdS, with curvature radius determined by the IR fixed point. Conversely, for $r_* \rightarrow -\infty$, the AdS background has unit curvature. Notice also that we have chosen the integration constant in the warp factor in such a way that (asymptotically) in the UV all the backgrounds become

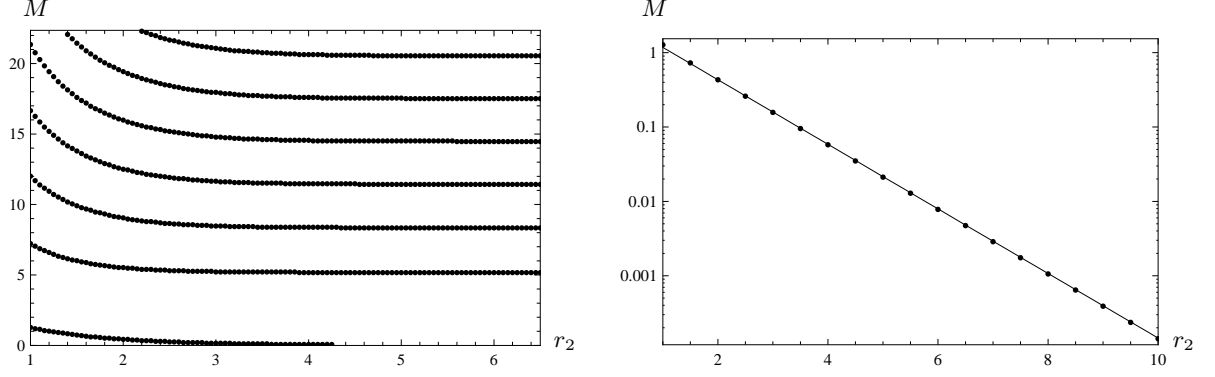


FIG. 6: Numerical results for Example B. Mass M of the lightest scalar fluctuations, keeping the IR cutoff fixed at $r_1 = 10^{-6}$ while varying the UV cutoff r_2 . The right panel shows a detail of the left one, with the line being the approximation $M \propto e^{-r_2}$.

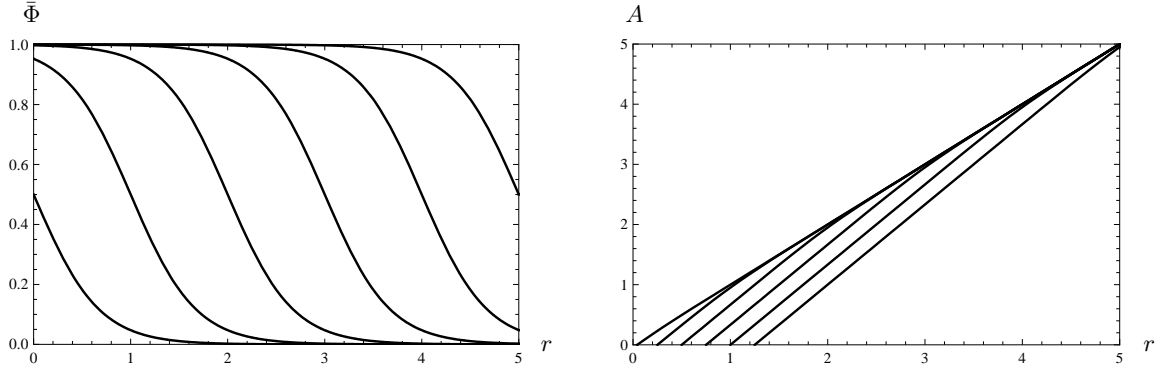


FIG. 7: Sampling of the functions determining the background in Example C, and used in the numerical analysis, plotted against the radial coordinate r . Plots obtained with $\Delta = 3$, $\Phi_I = 1$, $r_1 = 0$ and $r_2 = 5$. The backgrounds differ in the choice of r_* . Notice that the integration constant in the warp factor A is chosen so that the warp factor of all the different backgrounds agrees in the far UV ($r \rightarrow \infty$).

identical. This means that the same numerical choice for UV-quantities (such as r_2) corresponds to the same physical scale. This is not the case for IR-quantities (such as r_1). We will come back to this comment later on.

In Figure 8, we show the spectrum of masses M for the first few composite states as a function of r_* . The spectrum has been computed with $m_i^2 \rightarrow +\infty$. The different figures show the results for different values of Δ . Focusing on the heavy states, we observe the expected behavior: the spectrum of heavy states consists of an infinite tower of evenly spaced KK excitations. There is an artificial suppression of the masses as a function of r_* . This effect was already observed earlier on, when discussing the quadratic potential. Since below r_* the curvature is not unit, the fact that we chose the warp factors of different backgrounds to agree in the far UV means that the same value of r_1 yields a different physical scale. An alternative way of defining the IR cutoff is to make it be at the point where the warp factor A is equal to zero. This then ensures that the IR cutoff is always at the same energy scale, and in this sense it is perhaps more natural from a physical point of view. The resulting plots are shown in Figure 9 (for $\Phi_I = 1$) and Figure 10 (for $\Phi_I = 2$). There is nothing particularly deep about this, aside from suggesting that we need to exercise some caution when making quantitative statements relating physical scales to each other.

Much more interesting is what happens at the level of light states. The limits of r_* being small or large are associated with the physics of the two different fixed points ($\Phi = 0$ and $\Phi = \Phi_I$). For small values of r_* , the backgrounds are very similar to those considered in the case of quadratic superpotential in Sections IV A and V A, and the spectrum behaves qualitatively the same. For instance, in the case of $\Delta > 2$, there is a light scalar. On the other hand, the backgrounds with larger values of r_* only deviate away from the fixed point $\Phi = \Phi_I$ far in the UV, exhibiting walking behaviour from the IR cutoff up to the scale set by r_* . Therefore, in this case the light scalar has the interpretation of being the analog of the dilaton discussed in the context of walking technicolor. This picture is clearest for larger

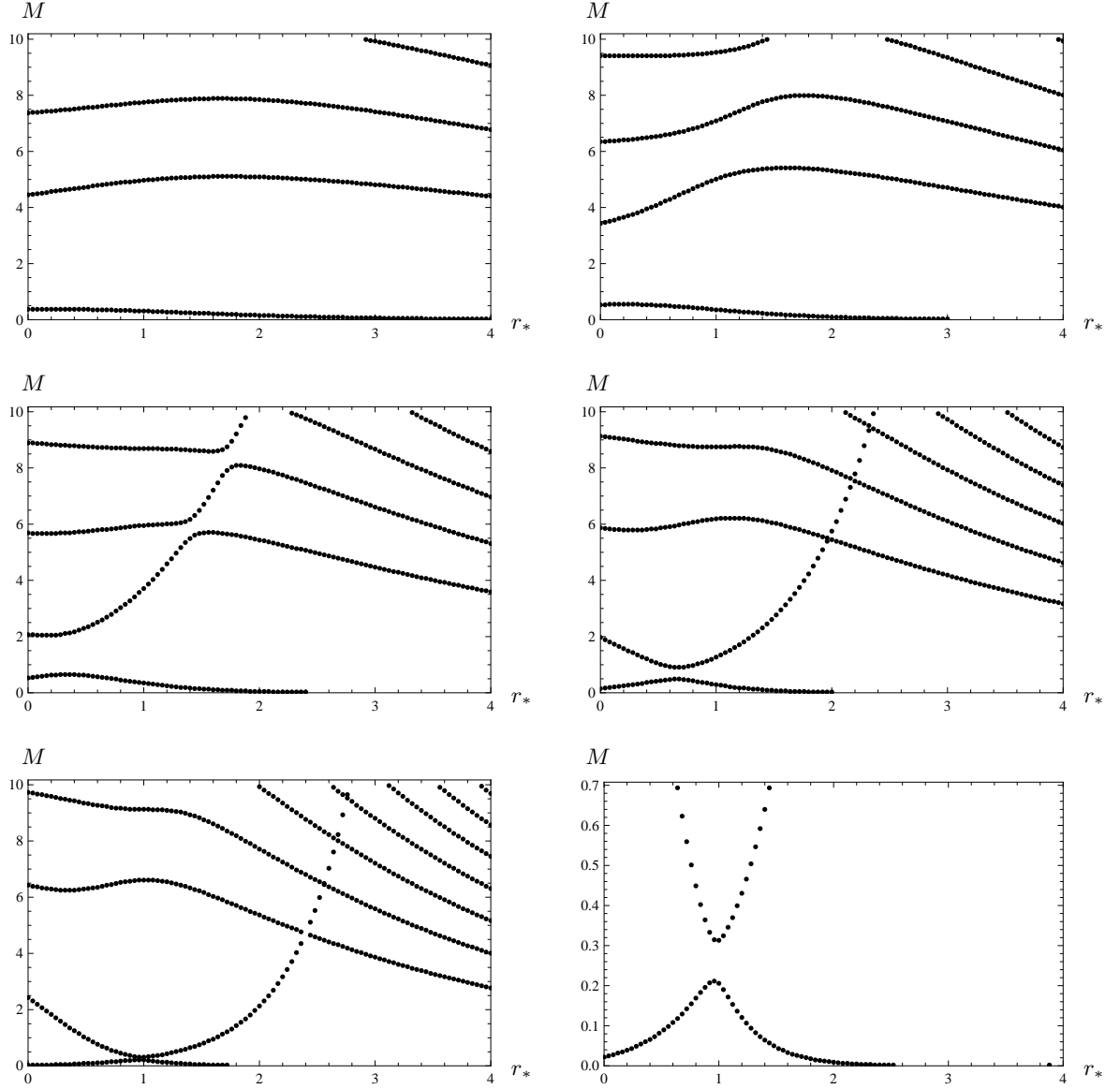


FIG. 8: Numerical results for Example C. Mass M of the lightest scalar fluctuations, for the choices $\Delta = 1, 1.5, 2, 2.5, 3, 3$ (left to right, top to bottom), $\Phi_I = 1$, $r_1 = 0$ and $r_2 = 5$, computed with backgrounds differing in the choice of r_* . The last panel shows a detail of the left one, with $\Delta = 3$.

values of Δ in which case an interesting crossing structure develops. The crossing structure makes it apparent that the nature of the lightest scalar changes radically as r_* is varied, being related to either of the two fixed points, for small or large values of r_* , respectively.

In Figure 11, the left panel illustrates that for larger values of Φ_I , the light state whose mass is suppressed by the length of the walking region (i.e. r_*) requires a longer such region to become light. The right panel of Figure 11 shows the UV cutoff dependence of the light states. As expected, the light scalar associated with the fixed point at $\Phi = 0$ has a mass that is suppressed by the UV cutoff r_2 in agreement with the results found for $\Delta > 2$ in Example A. The light scalar associated with the fixed point at Φ_I , however, is unaffected by the value of r_2 , caring only about r_* which acts as the UV cutoff for this state.

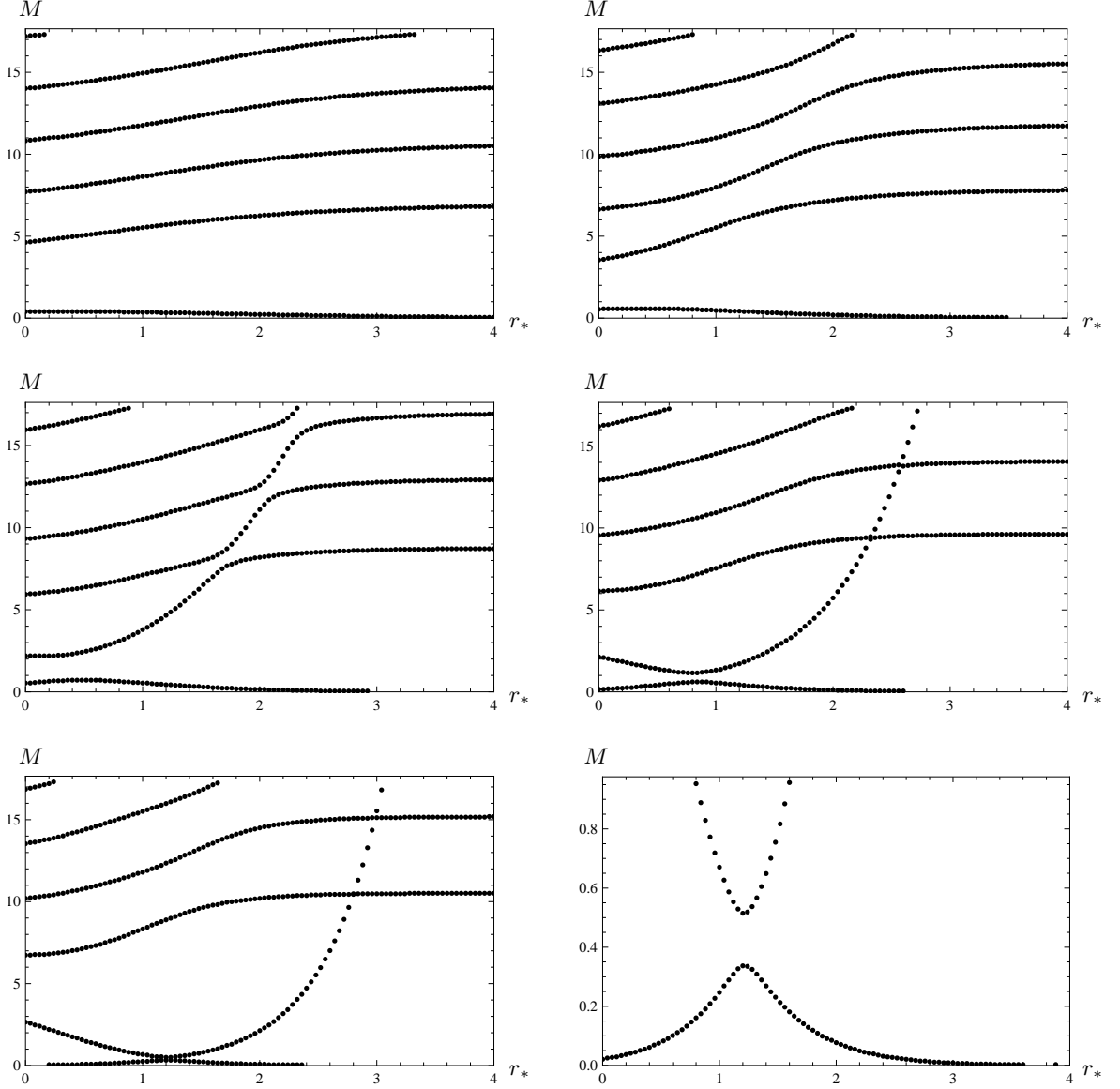


FIG. 9: Numerical results for Example C. Mass M of the lightest scalar fluctuations, for the choices $\Delta = 1, 1.5, 2, 2.5, 3, 3$ (left to right, top to bottom), $\Phi_I = 1$ and $r_2 = 5$, computed with backgrounds differing in the choice of r_* , and by choosing r_1 so that $A(r_1) = 0$. The last panel shows a detail of the left one, with $\Delta = 3$.

VI. FIELD THEORY INTERPRETATION, DISCUSSION AND GENERAL LESSONS

In this section, we discuss the physical meaning and implications of the examples we discussed in the paper. Before we start, we must remind the reader about the two main subjects of the paper. First of all, we are mostly interested in understanding under which conditions a strongly-coupled, (quasi) conformal theory admits anomalously light scalars in the spectrum. These can be the result of accidental cancellations, but more often are the result of the spontaneous breaking of approximate symmetries. Such symmetries can be internal (giving rise to pseudo-Goldstone bosons, such as the techni-pions and techni-axions of a technicolor model) or it may happen that scale-invariance is an approximate symmetry, in which case the light scalar is a dilaton.

Second, the framework within which we work is that of gauge-gravity dualities. What we did was to set up a very flexible formalism, that allows to study the four-dimensional spectrum obtained from a five-dimensional sigma-model of n scalars coupled to gravity, in the presence of UV and IR boundaries in the radial direction. The formalism exploits the diffeomorphism invariance of the five-dimensional theory in order to write the (linearized) equations for

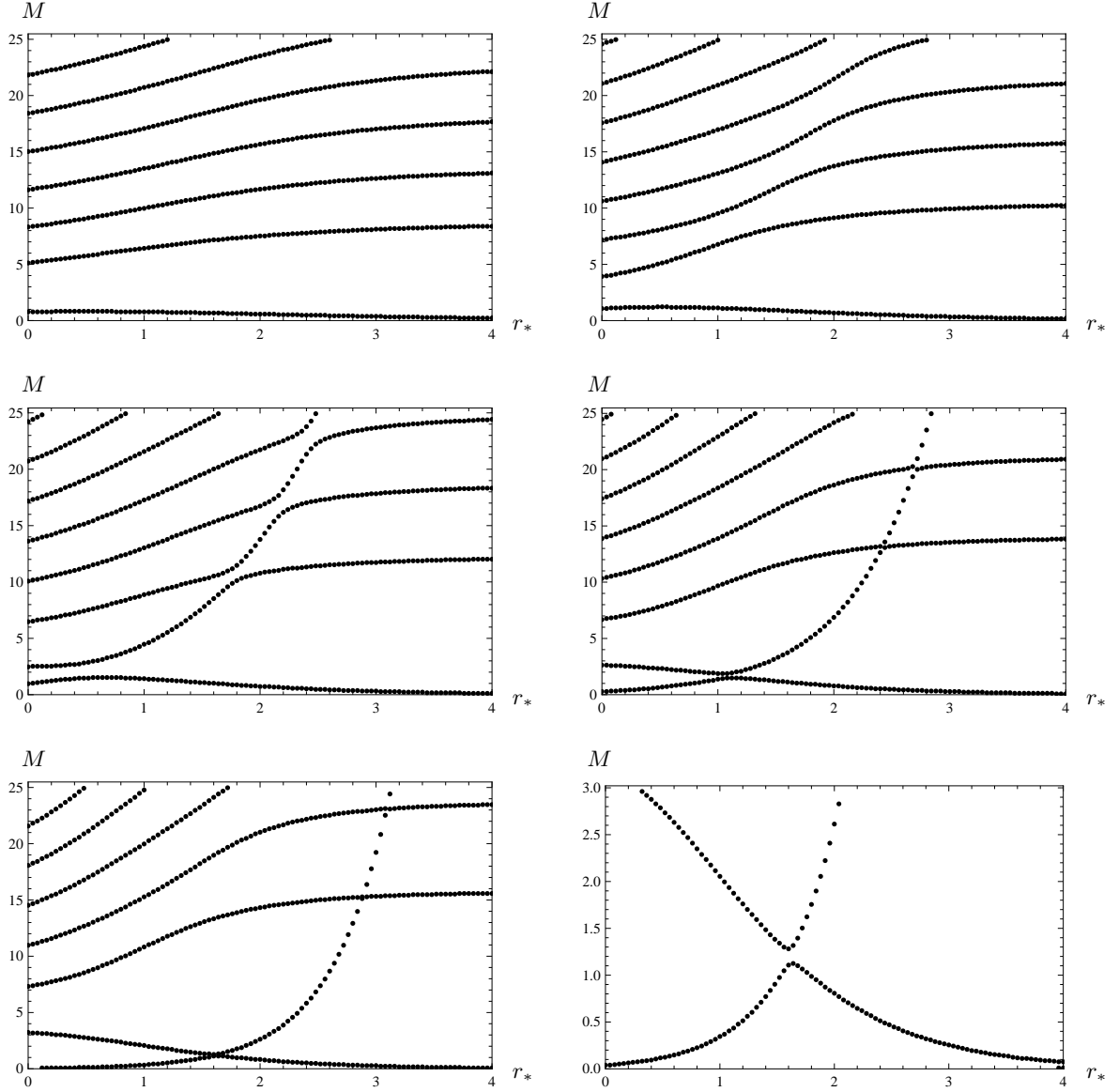


FIG. 10: Numerical results for Example C. Mass M of the lightest scalar fluctuations, for the choices $\Delta = 1, 1.5, 2, 2.5, 3$ (left to right, top to bottom), $\Phi_I = 2$ and $r_2 = 5$, computed with backgrounds differing in the choice of r_* , and by choosing r_1 so that $A(r_1) = 0$. The last panel shows a detail of the left one, with $\Delta = 3$.

the fluctuations and the boundary conditions directly in terms of physical states. This fact allows to reduce the complexity of the general problem to a set of n (coupled) equations involving only n scalar fields.

We applied this formalism to three examples. Up to now, we focused on the technical aspects, showing explicitly, on the basis of these three very simple examples, how the calculations are carried out, and what are the main results. All the examples involve only one scalar field, with trivial sigma-model metric, and all admit a description in terms of a superpotential. The latter is quadratic (in Example A), a simple hyperbolic function (in Example B), or cubic (in Example C).

For particular choices of the boundary potential, all of the examples can be made to contain at least one exactly massless mode. Its composition in terms of the fluctuations of the original scalar and gravity degrees of freedom always includes a component that couples to the trace of the four-dimensional stress-energy tensor. In other words, there is always a field that can be identified with the dilaton. The crucial task is then to identify under which conditions (on the bulk dynamics and on the boundary conditions) does this state stay light and keeps (at least at leading-order) the appropriate couplings in order to be identified as a physical dilaton. In order to do so effectively, one has also

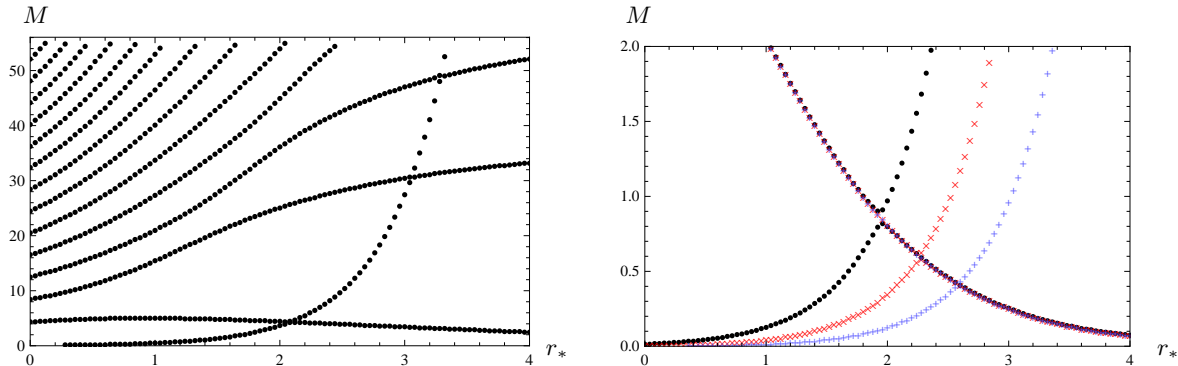


FIG. 11: Numerical results for Example C. The left panel shows the mass M of the lightest scalar fluctuations, for the choice $\Delta = 3$, $\Phi_I = 4$, and $r_2 = 5$, computed with backgrounds differing in the choice of r_* , and by choosing r_1 so that $\bar{A}(r_1) = 0$. The right panel shows the mass M of the lightest scalar fluctuations, for the choice $\Delta = 3$ and $\Phi_I = 2$, computed with backgrounds differing in the choice of r_* , by choosing r_1 so that $\bar{A}(r_1) = 0$, and varying the UV cutoff $r_2 = 6$ (black dots), $r_2 = 7$ (red \times), and $r_2 = 8$ (blue $+$).

to anticipate the effect that coupling the strongly-interacting sector (dual to the five-dimensional sigma-model) to an external weakly-coupled sector has on the spectrum. Hence, one has to make sure that no serious fine-tuning problems are present.

We saw that the boundary conditions are determined by the background solution, up to a certain amount of freedom in the choice of the quadratic m_i^2 terms. Special choices of the m_i^2 may yield very peculiar results for the spectrum. In particular, we saw that setting $m_2^2 = 0$ automatically implies that an exactly massless state is present. However, such a special choice is certainly the result of fine-tuning: as soon as coupling to an external sector is added, there is no reason to expect that such a choice is preserved. In general, (perturbative) loop corrections coming from this external sector are going to yield corrections to m_2^2 , which are UV-sensitive (divergent). For this reason, it is most interesting to see if a light state exists when $m_i^2 \rightarrow +\infty$, in which case one can be confident that no fine-tuning is hidden in our procedure. In doing so, one is also guaranteed to break any possible approximate global symmetries of the sigma-model (and of the dual strongly-interacting theory), so that if a light state exists, it cannot be due to such an internal symmetry. However, a word of caution is needed here: if for some physical reason the global symmetry of the internal sector happens to be also a global symmetry of the external, weakly-coupled sector of the full theory, then one has to treat the matrices m_i^2 appropriately.

Finally, a completely general comment. We said in the introduction that the physics of massive states cannot be completely universal, but rather it is necessarily sensitive to model-dependent details about the dynamics. If one had the exact dual of a well-known and established technicolor model that fits all the data, this observation would not matter, one could simply compute the spectrum in the gravity side of the correspondence, and conclude with the phenomenological implications of the results. This is unfortunately not the case, in part because no such a thing as a standard technicolor model exists, but also for a more subtle reason. Gauge/gravity dualities, in the context of dynamical electro-weak symmetry breaking, do provide precise and effective calculation techniques, in the sense that the results do not depend on uncalculable $O(1)$ coefficients, as is the case for four-dimensional estimates with strongly-coupled systems. But one faces the limitation that the gravity description is dual to models that are not precisely what one wants. In particular, all we are going to say is valid only in the strict large- N limit, and in most of the cases some amount of supersymmetry is present. Hence, in spite of the fact that one can get actual numbers, rather than order-of-magnitude estimates, for the relations between masses and couplings of the various states, one still needs to consider many different models, and try to understand the parametric dependences, rather than the actual numbers.

A. Quadratic superpotential

We start from Example A, in which case the superpotential is simply quadratic. It must be stressed that all that we are going to say is not restricted to this specific model, but rather it will hold also for any other model that is at least approximated by a set of controllable perturbations of a conformal theory, provided, in the whole region between the two cutoffs that we introduce, the vicinity to a fixed point controls the dynamics completely. We will hence use

some of the present considerations also in discussing Examples B and C, when appropriate.

The analysis of the spectrum of the scalar fluctuations has revealed the existence of a number of limits in which a parametrically light scalar emerges. Specifically, we found a light scalar if at least one of the conditions $\Delta \ll 1$, $m_2^2 \ll 1$, or $\Delta > 2$ holds. The first case has been discussed at length in the literature, and is known to be interpreted in terms of a quasi-marginal deformation of the CFT. The second case is the result of a fine-tuned choice, as we saw earlier on, and is hence of marginal interest. We hence focus on the last case, $\Delta > 2$.

When $\Delta > 2$, the standard dictionary of the AdS/CFT correspondence implies that an operator \mathcal{O} of dimension Δ has developed a VEV. In which case, one clearly expects a massless dilaton to be present. Because of the finite value of the UV cutoff r_2 that we use, otherwise subleading deformations (such as the insertion of multi-trace operators in the dual theory [35]) cannot be neglected, and result in a small mass for such a dilaton. The irrelevant nature of such deformations explains the suppression of the mass m_d^2 in Eq. (81) as a function of r_2 , and the fact that a massless state can be recovered in the $r_2 \rightarrow +\infty$ limit.

In order to make this more quantitative, let us assume that the dual theory is such that $\dim \mathcal{O}^2 = 2 \dim \mathcal{O}$ (e. g. in the large- N limit). In the study performed in the body of the paper, we take the limit $r_2 \rightarrow +\infty$ by holding the VEV Φ_1 fixed. A simple toy-model description of the dual effective potential, due to the multi-trace deformations that break conformal invariance for finite r_2 , can be approximately given by $\Lambda^{4-2\Delta} (\mathcal{O} - v)^2$, where v is the VEV in the field theory, and the scale Λ is related to the UV cutoff. For $\Delta > 2$, this effective potential is suppressed by the UV scale, because the operator \mathcal{O}^2 is irrelevant (together with any even higher order \mathcal{O}^n correction), hence explaining our result that $m_d^2 \propto e^{-2(\Delta-2)r_2}$ from Eq. (81).

B. Example B

The second example we considered can be thought of as a completion, in the context of Type-IIB, of the case with quadratic potential in Example A, for $\Delta = 3$ (and $\Phi_1 = 1$) and for $\Delta = 1$ (and $\Phi_1 = \sqrt{3}$). This is nice for several reasons. First of all, it means that the superpotential is known beyond the quadratic level, and hence can be used with some degree of confidence also far away from the UV fixed point. Second, if one chooses $d\Omega_5$ in such a way that for $\Phi = 0$ it yields the metric on the five-dimensional sphere, one has a precise mapping in terms of deformations of $\mathcal{N} = 4$ super-YM. In the $\Delta = 3$ case, we are enforcing a non-trivial gaugino condensate [31], while for $\Delta = 1$ we are giving a mass to the fermions. Yet, the presence of a singularity in the IR of the background means that this is not the complete dual to the four-dimensional field theory, but that some ingredient is missing, so that the physics of the four-dimensional theory is well-captured only away from the singularity. In practice, this means that the truncated system yielding our very simple superpotential is actually too simple: it does not capture some non-trivial properties of the strong dynamics of $\mathcal{N} = 4$, taking place very far from the UV fixed-point. This is all well known, and discussed in the literature.

It is therefore interesting to understand what happens when computing the spectrum using our algorithmic procedure, which implies adding cutoffs both in the UV and in the IR. There is not much to say about the $\Delta = 1$ case. This just provides a very nice cross-check, showing how the regulator procedure we use yields the correct results, in a simple case in which other arguments can be used to discuss the spectrum without introducing any regulators.

We hence focus here on the $\Delta = 3$ case, which we extensively studied numerically. Since $\Delta > 2$, for finite UV cutoff the interpretation is that we add an irrelevant deformation to a theory and enforce a VEV. Provided the IR cutoff r_1 is far away from the singularity, the VEV is spontaneously breaking scale invariance, as in Example A. However, as r_1 is chosen to be close to the singularity, there is no obvious sense in which the theory is still close to a conformal fixed-point, and hence no a priori reason to expect the light state to persist. It is hence very interesting, and somewhat surprising, that it does. Notice by contrast that the spectrum of heavy states is shifted, to testify of the fact that the singularity is not a negligible correction to Example A. Even more, remember that the study of the Wilson loops we briefly sketched shows a very dramatic effect, to the point that the string probe cannot even approach the region in the immediate proximity of the singularity.

The fact that the IR is badly singular clearly signals that the strong dynamics of $\mathcal{N} = 4$ involves other non-trivial effects, which are not captured by this simplest truncated model, as we said earlier. Yet, the boundary action localized at $r = r_1$ acts as an IR regulator, which effectively removes from the calculation of the scalar spectrum the pathological effects of the IR singularity. The resulting spectrum has all the sensible features expected in a healthy field theory. It would be very nice to know if (and to what degree) the spectrum we computed is in quantitative agreement with what is obtained in a modification of the model such that the IR singularity is resolved. In particular, we found the presence of a very light state (which did not exist for $\Delta = 1$), and one should test whether this state is still present in the more complete analysis, rather than being an artifact due to the combined effects of the bad singularity and the IR regulator.

C. Cubic superpotential

The field theory motivation for studying this example is that one might be interested in studying four-dimensional models in which the RG-flow of a confining (UV-complete) theory happens to come very close to a non-trivial fixed point at intermediate energies. One such example of phenomenological relevance is the class of models that embed walking technicolor into extended technicolor, in order to explain in a unified picture the three (superficially conflicting) requirements of generating large masses for standard-model fermions while at the same time suppressing new-physics contributions to FCNC processes and preserving the UV-completeness of the theory.

There is no known example of a dynamical model the exact gravity dual of which has all the features required in the walking technicolor framework. There exist models that reproduce the flow of a confining theory, and there are models that describe the flow between two fixed points, but there are no models in which the IR fixed point is only approximate, and ultimately the theory confines. Also, models yielding the flow between two fixed points have a tendency to be very complicated, such as the Pilch-Warner dual of the flow from $\mathcal{N} = 4$ to the Leigh-Strassler fixed point, for which the actual background is known only numerically (and within this framework, flows that approach the IR fixed-point but do not reach it are described by backgrounds which are badly singular).

Yet, it is of general interest to know whether, in such a theory, it is at all possible that a light dilaton is present. Notice that the answer to this question is not obvious: even when considering the exact flows between two exact fixed points, (in which case both the far-UV and deep-IR effective descriptions are provided by CFTs) the theory as a whole is not scale-invariant: there exists a physical scale (connected to the ρ^* in our study) that separates the regimes in which either of the two CFTs provides a sensible approximation for the physics. So, the actual mass of the lightest scalar (the would-be dilaton) will in general depend on this scale, on the two CFTs living at the fixed points (the spectrum of dimensions), and on the specific properties of the flow.

The study we performed shows that indeed there is a light scalar, under quite general conditions, and that its mass depends on the dimension Δ and on ρ^* . Let us start from the dependence on the dimensions. At the IR fixed point, all the (active) scalars must correspond to irrelevant deformations (otherwise the flow could not reach such a fixed point). The only important distinction comes from the value of the dimension Δ , which characterizes the flow near the UV fixed point.

What is most remarkable, is that when $\Delta > 2$ the dependence on r^* of the mass of the lightest state is not a monotonic function. There exists an actual maximum of the mass, obtained for non-extreme values of r^* . The reason for this is that when r^* is so large, or so small, that the theory is effectively always very close to one of the two fixed points, the mass of the lightest state is suppressed parametrically, because the theory is effectively very close to conformal. For intermediate values of r^* , the theory is not well described by a small departure from a CFT, and as a result a non negligible mass is generated for the lightest state. However, the r^* dependence of such mass must interpolate between the two extremal values of r^* (the UV and IR cutoffs), near which the mass practically vanishes. Hence, there is a maximum for such mass. It is curious to notice that numerically such a maximum is (at least in our examples) still significantly suppressed in respect to the typical scale of the heavy states, although the actual physical relevance of such a fact is questionable.

VII. CONCLUSIONS AND OUTLOOK

We conclude the paper by summarizing our main results, by critically reviewing the limitations of this approach and by suggesting a set of physically interesting applications.

A. The algorithm

Let us first summarize the algorithm to be used to compute the spectrum. We start with a five-dimensional sigma-model consisting of n scalars coupled to gravity. For simplicity, suppose that a superpotential W is known, and furthermore that there is no obstruction to taking $m_i^2 \rightarrow \infty$. The spectrum of scalar bound states can then be computed by applying the following steps.

- Write the action in the form of Eq. (29), with the bulk dynamics of the n scalars in the form of Eq. (30), and the background metric in the form of Eq. (16).
- Introduce a UV and an IR cutoff, by writing boundary actions as in Eq. (31) and Eq. (32).
- Determine the background, by solving Eq. (39) and Eq. (40), subject to the boundary conditions in Eq. (35) and Eq. (36).

- Obtain the spectrum by solving Eq. (50), and then identify which values of $q^2 = \square$ allow to satisfy the boundary conditions in Eq. (53).
- If possible, and physically meaningful, take the limits $r_2 \rightarrow +\infty$ and $r_1 \rightarrow r_0$, where r_0 is the end-of-space in the IR.

If the superpotential is not known, or it is not legitimate to take $m_i^2 \rightarrow \infty$, all the necessary changes to be implemented in this procedure are explained in the body of the paper. It is also implicit that one should familiarize oneself with the notation, which is explained in detail in Section II, where all the relevant information is provided explicitly.

B. Limitations of the algorithm

The algorithm we identified fails, or needs to be partially extended, in the following cases.

- The bulk action contains terms with more than two derivatives of the scalars and/or the metric. In this case, the whole procedure has to be rethought from scratch.
- The dependence of the two-point functions on the UV cutoff requires introducing q^2 -dependent boundary terms in the UV, in order to regularize the theory and remove the UV cutoff itself. If these terms are polynomial, they just result in a comparatively harmless modification of Eq. (53), to be dealt with via holographic renormalization. If they are non-polynomial, then the whole physical meaning of the spectrum becomes questionable, and the best thing one can do is to consider the dual theory as some phenomenological model with a physical cutoff r_2 , which cannot be removed.
- There are exactly flat directions in the supergravity potential, connected with moduli of the field theory. In this case, there are exactly massless states which have nothing to do with the dilaton, and at the technical level the \square^{-1} operator appearing in many equations is badly defined. One should find a (model-dependent) way to overcome this difficulty, either by adding a perturbation that lifts the flatness of the potential, or by further truncating the sigma-model in such a way as to decouple the potentially problematic (inactive) fluctuations.
- In the IR, the space ends in a naked singularity at r_0 , and the geometry near the singularity is so bad that (super)gravity cannot be trusted. In this case, one has to keep the IR cutoff $r_1 > r_0$, and firm physical conclusions cannot be drawn in full generality, until a resolution of the singularity within a more general sigma-model is found. However, note that in the one example that we studied where a naked singularity is present, i.e. Example B, our algorithm in fact yields finite results.
- The IR is not singular, however the end of space is known to be described by extended objects that go beyond the (super)gravity approximations. Again, in this case one can only keep $r_1 > r_0$, using a cutoff that chops off the space at a scale where the (super)gravity description still holds. It is not known (and it would be interesting to know) whether a relation between the spectrum computed with the present algorithmic procedure and the actual physical spectrum computed by fluctuating the whole background exists.

C. Physics lessons

The main physical motivation of this work, as explained in the introduction, is to understand what kind of (confining) strongly-coupled theories yield potentially light scalars in the spectrum, one (linear-combination) of which has the couplings of a four-dimensional dilaton. None of the examples we considered is the exact dual of such a theory, and yet in all the examples a set of very general lessons appears to emerge coherently. We summarize here these results.

When the dynamics of the theory is well described in terms of the properties of the RG fixed points and their proximity, a light scalar appears to be present, irrespectively of model-dependent details, provided the dimension Δ (defined in the body of the paper) satisfies either $\Delta \geq 2$ or $\Delta \ll 1$. In the latter case, this is the well-known fact that a very small Δ means that a (quasi-)marginal operator is deforming the CFT, hence introducing a parametrically small explicit breaking of conformal symmetry. This is true even in the Higgs sector of the Standard Model, provided all the couplings are very weak, and is hence not particularly interesting. A lesser known fact is that this is true also when $\Delta \geq 2$. Provided factorization holds, one can interpret this particular result in terms of the spontaneous breaking of scale invariance via the VEV of an operator \mathcal{O} of dimension $\Delta \geq 2$. The fact that the (double-trace) deformation \mathcal{O}^2 is irrelevant ensures that the light scalar present has a mass suppressed by the UV cutoff.

However, one has to exert some caution in using these results. First of all, they hold only in the limit where factorization is exact, and no non-trivial operator mixing is present. Most important, they depend on what is meant by the dynamics being well described in terms of its properties in proximity of the fixed-points. For example, what about QCD (or, maybe better, Yang-Mills)? It is well known that QCD admits a trivial UV fixed point, and that the departure from it is induced by a quasi-marginal deformation. And yet, there is no light scalar state in the glueball spectrum. The reason is that the RG flow goes very far away from the fixed point, and at the confinement scale there is no sense in which the theory is still close to it. Indirectly, this also means that it is very unlikely that the physics near the confinement scale is controlled by a non-perturbative fixed point: in this case, as we saw, in spite of the fact that the whole theory is not scale-invariant, a light state would still be expected, which is not the case for QCD/YM.

A set of minor caveats emerged during the technical calculations. Summarizing their implications: in the cases where we identified the presence of a light state in the spectrum, one should not immediately conclude that such a light state is there in general, but should then ask whether the process of coupling the strongly coupled sector to external observable sectors still preserves this result. The whole problem of holographic renormalization is implied.

Finally, we conclude with some important comments about walking technicolor and similar theories. In this case, one is interested in a model where the RG flow starts near a fixed point, evolves towards a different fixed-point, but then never reaches the latter, and ultimately confines, with the formation of non-trivial condensates that break the global symmetries of the theory. What did we learn about this scenario? While we do not have the gravity dual of a walking theory, and hence a firm conclusion should be postponed to a future study, yet by combining the results of Examples A, B and C we obtained a set of important indications of the presence of a very light state in some cases of walking dynamics.

Let us summarize here all the elements leading to this conclusion. In studying Example C, we considered the case where the theory is well-approximated by the dynamics near the UV fixed point for $r \gg r_*$, and by the dynamics near the IR fixed-point for $r \ll r_*$. We chopped off the space in the IR, assuming this to be the end-of-space of the model. When r_* is small, effectively the theory is never really well described by the IR fixed point, and for all practical purposes, this is not a walking theory. When r_* is large enough that a sizable range $0 < r < r_*$ of the fifth-dimension can be thought of as describing walking, something interesting happens. There is always a light state, that has a natural interpretation in terms of a dilaton. The mass of such state is suppressed by r_* : the larger r_* is, the lighter the dilaton. Ultimately, this result agrees with the expectations from Example A, in the sense that if one expands the background near the IR fixed point, the resulting Δ is always going to be large. From the field theory point of view, this must be the case: if the RG flow was governed by a relevant coupling, it could not approach the fixed point, which implies that by expanding near the IR fixed point one must find a large value for Δ . As a result, r_* represents the scale of explicit breaking of scale invariance, and enters the effective description near the IR fixed point by suppressing the coefficient of the irrelevant operator \mathcal{O}^2 inducing the explicit breaking itself. This is very interesting for practical purposes: it means that one can write the mass of the light scalar in terms of r_* and $\Delta > 2$, hence providing a functional relation between the walking scale associated with r_* and the mass $m_d^2 \propto e^{-2(\Delta-2)r_*}$.

All of this means that the very fact that the theory walks (in the sense of coming very close to a IR fixed point) implies that a potentially light dilaton is present, and that the UV scale at which walking ends effectively suppresses its mass. One has then to ask what is the effect on the mass due to the fact that in the IR the theory confines. Example B yields some very interesting information, by looking at the $\Delta = 3$ and $\Delta = 1$ cases, and comparing the results with Example A. Even when the IR ends in a singularity, the mass of the lightest scalar is in substantial agreement with what was found in Example A (notice that this is not the case for the heavy KK state, for which the shift in mass is significant). The most important parameter appears to be Δ . If $\Delta > 2$, effectively the confining behavior can be attributed to the formation of a condensate that breaks spontaneously scale invariance, with explicit breaking accounting only for a small effect. On the contrary, for $\Delta < 2$ no light state exists (unless $\Delta \ll 1$ which we already commented about), and scale invariance is broken explicitly. In a realistic walking technicolor theory, the condensate that takes the theory away from the IR fixed point is presumably related to the electro-weak scale. On the basis of phenomenological considerations it is usually believed that the dimension of the chiral-symmetry breaking condensate must be $2 \leq \Delta \leq 3$, which would imply the existence of a light dilaton in the spectrum.

This opens a possibility that in certain walking technicolor theories, in which the walking scale is parametrically higher than the electroweak scale, a parametrically light scalar is present in the spectrum. This light scalar should then be interpreted as a light dilaton, and hence have couplings that are very similar to those of the SM light Higgs. Hence, the discovery of such a light scalar at the LHC might be interpreted as a first indication in favor of a walking technicolor origin for electroweak symmetry breaking. Only the (non)detection of somewhat heavier resonances, in the TeV region, would allow to solve this possible ambiguity in experimental signals shared by the weakly-coupled Higgs models and the strongly coupled walking models. It would be very nice to test this conclusion for an explicit, complete model of the gravity dual of a walking theory, hence putting this line of arguments of firm grounds.

D. Outlook

The whole machinery we put in place and summarized here has its natural application in the study of the spectrum of a confining theory the full dynamics of which is well-captured within a five-dimensional sigma-model which is obtained by consistently truncating a fundamental, UV-complete theory. No phenomenologically useful such constructions exists yet, and hence an obvious direction for further research is to try to identify such a model. Doing so, and hence exploiting in full the potential of gauge-gravity dualities, has the great technical advantage that very non-trivial properties of the dual theory can be computed, and the algorithmic procedure we provided renders the extraction of the spectrum a relatively harmless technical matter.

In particular, our comments on walking dynamics and walking technicolor are not supported by very robust arguments, but rather based on circumstantial evidence emerging from a set of toy models, which we chose mostly on the basis of their simplicity. Yet, our work shows that it is important to find complete duals of walking theories within superstring theory, and that the study of the spectrum of such models could yield very non-trivial results of utmost phenomenological and theoretical importance.

From a more formal point of view, and in the short-term, a large number of interesting questions have been left open by this study, and require applying the procedure we outlined to less ambitious five-dimensional backgrounds. We mostly concentrated here on semi-realistic toy-models. It would be very interesting to apply all of the above to the many well-known backgrounds that exist in the literature, and whose dual four-dimensional theories are well-understood. One could then compare the results obtained using the IR and UV regulators to the results obtained with other techniques, both on the field-theory side and on the gravity side of the correspondence. This would yield important tests of the correctness of this procedure, and might help to shed light on the physical implications for models where the field theory is understood only in part. Also, it would be interesting to implement all of the above within the systematic program of holographic renormalization.

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Appendix A: Equations of Motion in the ADM formalism

Here, and in the following appendices, we derive all the equations of motion, and the boundary conditions for the background as well as the fluctuations. Much of the notation is the same as that of [25].

Starting with the action of the non-linear sigma model with boundary terms (where in the body of the paper $d = 4$)

$$S = \int d^d x dr \left[\sqrt{-g} \left(\frac{R}{4} - \frac{1}{2} G_{ab} \partial \Phi^a \partial \Phi^b - V(\Phi) \right) + \sqrt{-\tilde{g}} \delta(r - r_1) \left(\lambda_{(1)}(\Phi) - \frac{K}{2} \right) - \sqrt{-\tilde{g}} \delta(r - r_2) \left(\lambda_{(2)}(\Phi) - \frac{K}{2} \right) \right], \quad (\text{A1})$$

we derive the equation of motion for the scalars

$$\nabla^2 \Phi^a + \mathcal{G}_{bc}^a g^{MN} (\partial_M \Phi^b) (\partial_N \Phi^c) - V^a = \sum_i \sqrt{\tilde{g}} g^{-1} \lambda_{(i)}^a \delta_i \quad (\text{A2})$$

and Einstein's equations

$$-R_{MN} + 2G_{ab} (\partial_M \Phi^a) (\partial_N \Phi^b) + \frac{4}{d-1} g_{MN} V = 2 \sum_i \sqrt{\tilde{g}} g^{-1} \left(\hat{g}_{MN} - \frac{g^{KL} \hat{g}_{KL}}{d-1} g_{MN} \right) \lambda_{(i)} \delta_i, \quad (\text{A3})$$

where $\hat{g}_{\mu\nu} = g_{\mu\nu}$, and $\hat{g}_{r\mu} = \hat{g}_{rr} = 0$ (where the indices μ and ν run over the d -dimensional space-time), and tilde is used to refer to d -dimensional quantities.

We will now rewrite these equations of motion using the ADM formalism. We start by writing the metric on the form

$$g_{MN} = \begin{pmatrix} \tilde{g}_{\mu\nu} & n_\nu \\ n_\mu & n_\mu n^\mu + n^2 \end{pmatrix}, \quad (\text{A4})$$

where comparing to the notation used in the body of the paper, we have that $n = 1 + \nu$ and $n_\mu = \nu_\mu$. The inverse metric is given by

$$g^{MN} = \frac{1}{n^2} \begin{pmatrix} n^2 \tilde{g}^{\mu\nu} + n^\mu n^\nu & -n^\nu \\ -n^\mu & 1 \end{pmatrix}. \quad (\text{A5})$$

The tangent vectors X_μ^M are given by $X_\mu^r = 0$ and $X_\mu^\nu = \delta_\mu^\nu$. We have a normal vector $N_M = (0, n)$, $N^M = n^{-1}(-n^\mu, 1)$. The second fundamental form is

$$\mathcal{K}_{\mu\nu} = n\Gamma_{\mu\nu}^r = -\frac{1}{2n}(\partial_r g_{\mu\nu} - \tilde{\nabla}_\mu n_\nu - \tilde{\nabla}_\nu n_\mu), \quad (\text{A6})$$

where we use the notation of [25] (which only differs slightly from the one used in the body of this paper i.e. $K_{MN} \equiv \nabla_M N_N$), so that expressions can be easily compared. One can derive the following relations

$$\begin{aligned} \Gamma_{\mu\nu}^\sigma &= \tilde{\Gamma}_{\mu\nu}^\sigma - \frac{n^\sigma}{n} \mathcal{K}_{\mu\nu}, \\ \Gamma_{\mu r}^r &= \frac{1}{n} \partial_\mu n + \frac{n^\nu}{n} \mathcal{K}_{\mu\nu}, \\ \Gamma_{\mu r}^\sigma &= \tilde{\nabla}_\mu n^\sigma - \frac{n^\sigma}{n} \partial_\mu n - n \mathcal{K}_{\mu\nu} \left(g^{\nu\sigma} + \frac{n^\nu n^\sigma}{n^2} \right), \\ \Gamma_{rr}^r &= \frac{1}{n} (\partial_r n + n^\nu \partial_\nu n + n^\mu n^\nu \mathcal{K}_{\mu\nu}), \\ \Gamma_{rr}^\sigma &= \partial_r n^\sigma + n^\mu \tilde{\nabla}_\mu n^\sigma - n \tilde{\nabla}^\sigma n - 2n \mathcal{K}_\mu^\sigma n^\mu - n^\sigma \Gamma_{rr}^r. \end{aligned} \quad (\text{A7})$$

Finally, we are ready to write down the expressions for the equations of motion using the quantities defined above. The equation of motion for the scalars becomes

$$\begin{aligned} &\left\{ \partial_r^2 - 2n^\mu \partial_\mu \partial_r + n^2 \tilde{\nabla}^2 + n^\mu n^\nu \tilde{\nabla}_\mu \partial_\nu - (n \mathcal{K}_\mu^\mu + \partial_r \ln n - n^\mu \partial_\mu \ln n) \partial_r + \right. \\ &\quad \left[n \tilde{\nabla}^\mu n - \partial_r n^\mu + n^\nu \tilde{\nabla}_\nu n^\mu + n^\mu (n \mathcal{K}_\nu^\nu + \partial_r \ln n - n^\nu \partial_\nu \ln n) \right] \partial_\mu \Big\} \Phi^a + \\ &\quad \mathcal{G}_{bc}^a \left[(\partial_r \Phi^b) (\partial_r \Phi^c) - 2n^\mu (\partial_\mu \Phi^b) (\partial_r \Phi^c) + \right. \\ &\quad \left. (n^2 \tilde{g}^{\mu\nu} + n^\mu n^\nu) (\partial_\mu \Phi^b) (\partial_\nu \Phi^c) \right] - n^2 G^{ab} \frac{\partial V}{\partial \Phi^b} = n^2 \sum_i \sqrt{\tilde{g} g^{-1}} \lambda_{(i)}^a \delta_i. \end{aligned} \quad (\text{A8})$$

Einstein's equations separate into normal, mixed, and tangential components, obtained by projecting with $P^{MN} = N^M N^N - \tilde{g}^{\mu\nu} X_\mu^M X_\nu^N$, $P_\mu^{MN} = N^M X_\mu^N$, and $P_{\mu\nu}^{MN} = X_\mu^M X_\nu^N$, respectively. The normal component reads

$$\begin{aligned} &(n \mathcal{K}_\nu^\nu) (n \mathcal{K}_\mu^\mu) - (n \mathcal{K}_\mu^\mu)^2 + n^2 \tilde{R} - 4n^2 V + 2G_{ab} \left[(\partial_r \Phi^a) (\partial_r \Phi^b) - \right. \\ &\quad \left. 2n^\mu (\partial_\mu \Phi^a) (\partial_r \Phi^b) + (n^\mu n^\nu - n^2 \tilde{g}^{\mu\nu}) (\partial_\mu \Phi^a) (\partial_\nu \Phi^b) \right] = 4n^\mu n_\mu \sum_i \sqrt{\tilde{g} g^{-1}} \lambda_{(i)} \delta_i. \end{aligned} \quad (\text{A9})$$

In deriving this expression, the following relations are useful. We have that $P^{\mu\nu} = \frac{n^\mu n^\nu}{n^2} - \tilde{g}^{\mu\nu}$, $P^{r\mu} = -\frac{n^\mu}{n^2}$, and $P^{rr} = \frac{1}{n^2}$, so that $P^{MN} g_{MN} = 1 - d$ and $P^{MN} \hat{g}_{MN} = \frac{n^\mu n_\mu}{n^2} - d$. Furthermore, $g^{MN} \hat{g}_{MN} = \frac{n^\mu n_\mu}{n^2} + d$. The mixed component is given by

$$\begin{aligned} &\partial_\mu (n \mathcal{K}_\nu^\nu) - \tilde{\nabla}_\nu (n \mathcal{K}_\mu^\nu) - n \mathcal{K}_\nu^\nu \partial_\mu \ln n + n \mathcal{K}_\mu^\nu \partial_\nu \ln n - \\ &2G_{ab} (\partial_r \Phi^a - n^\nu \partial_\nu \Phi^a) \partial_\mu \Phi^b = -2n_\mu \sum_j \sqrt{\tilde{g} g^{-1}} \lambda_{(j)} \delta_j. \end{aligned} \quad (\text{A10})$$

and the tangential component is

$$\begin{aligned}
& -\partial_r(n\mathcal{K}_\nu^\mu) + n^\sigma \tilde{\nabla}_\sigma(n\mathcal{K}_\nu^\mu) + n\mathcal{K}_\nu^\mu(n\mathcal{K}_\sigma^\sigma + \partial_r \ln n - n^\sigma \partial_\sigma \ln n) + n\tilde{\nabla}^\mu \partial_\nu n + \\
& n\mathcal{K}_\sigma^\mu \tilde{\nabla}_\nu n^\sigma - n\mathcal{K}_\nu^\sigma \tilde{\nabla}_\sigma n^\mu - n^2 \tilde{R}_\nu^\mu + 2n^2 G_{ab}(\tilde{\nabla}^\mu \Phi^a)(\partial_\nu \Phi^b) + \frac{4n^2 V}{d-1} \delta_\nu^\mu = \\
& -\frac{2\delta_\nu^\mu}{d-1}(n^2 + n^\sigma n_\sigma) \sum_l \sqrt{\tilde{g}g^{-1}} \lambda_{(l)} \delta_l.
\end{aligned} \tag{A11}$$

At zeroth order in the fluctuations, the equations of motion for the scalars are

$$\bar{\Phi}''^a + dA' \bar{\Phi}'^a + \mathcal{G}_{bc}^a \bar{\Phi}'^b \bar{\Phi}'^c - V^a = \sum_i \lambda_{(i)}^a \delta_i, \tag{A12}$$

and Einstein's equations yield

$$\begin{aligned}
& d(1-d)A'^2 + 2G_{ab} \bar{\Phi}'^a \bar{\Phi}'^b - 4V = 0, \\
& A'' + dA'^2 + \frac{4}{d-1}V = -\frac{2}{d-1} \sum_i \lambda_{(i)} \delta_i.
\end{aligned} \tag{A13}$$

Writing the background as

$$\begin{aligned}
\bar{\Phi}'(r) &= (\Theta(r-r_1) - \Theta(r-r_2)) \hat{\Phi}'(r), \\
A'(r) &= (\Theta(r-r_1) - \Theta(r-r_2)) \hat{A}'(r),
\end{aligned} \tag{A14}$$

we obtain the boundary conditions

$$\begin{aligned}
\bar{\Phi}'^a \Big|_{r_i} &= \lambda_{(i)}^a \Big|_{r_i}, \\
A' \Big|_{r_i} &= -\frac{2}{d-1} \lambda_{(i)} \Big|_{r_i},
\end{aligned} \tag{A15}$$

where we have dropped the hats.

Appendix B: Linearized Equations of Motion

Let us now expand the equations of motion in fluctuations of the metric and the scalar fields to linear order. As explained in Section IID, we can work in a gauge where $n^\mu = \nu^\mu = 0$. Furthermore, since we want to consider only spin-0 fluctuations, we can put $\epsilon^\mu = h^{TT\mu}_\nu = 0$. This leaves us with the fluctuation variables (φ, ν, h, H) . Thus, we expand equations (A8), (A9), (A10), and (A11) using the following rules:

$$\begin{aligned}
\Phi^a &= \bar{\Phi}^a + \varphi^a, \\
n &= 1 + \nu, \\
h_\nu^\mu &= \frac{\delta_\nu^\mu}{d-1} h + \frac{\partial^\mu \partial_\nu}{\square} H.
\end{aligned} \tag{B1}$$

In doing so, we will make use of the relations

$$\begin{aligned}
\sqrt{\tilde{g}g^{-1}} &= 1 - \nu, \\
n\mathcal{K}_\nu^\mu &= -A' \delta_\nu^\mu - \frac{\delta_\nu^\mu}{2(d-1)} h' - \frac{1}{2} \frac{\partial^\mu \partial_\nu}{\square} H', \\
R_\nu^\mu &= -\frac{\delta_\nu^\mu}{2(d-1)} e^{-2A} \square h - \frac{d-2}{2(d-1)} e^{-2A} \partial^\mu \partial_\nu h, \\
R &= -e^{-2A} \square h,
\end{aligned} \tag{B2}$$

which are true to first order in the fluctuations. The linearized equation of motion for the scalar becomes

$$\begin{aligned}
& \partial_r^2 \varphi^a + e^{-2A} \square \varphi^a + dA' \partial_r \varphi^a + 2\mathcal{G}_{bc}^a \bar{\Phi}'^b \partial_r \varphi^c + \partial_d \mathcal{G}_{bc}^a \bar{\Phi}'^b \bar{\Phi}'^c \varphi^d - \frac{\partial V^a}{\partial \Phi^c} \varphi^c + \\
& \bar{\Phi}'^a \left(\frac{d}{2(d-1)} \partial_r h + \frac{1}{2} \partial_r H - \partial_r \nu \right) - 2V^a \nu = \sum_i \delta_i (\lambda_{(i)}^a \nu + \partial_c \lambda_{(i)}^a \varphi^c).
\end{aligned} \tag{B3}$$

At first order in the fluctuations, the normal component of Einstein's equations gives

$$4\bar{\Phi}'_a(\mathcal{D}_r\varphi^a) - 4V_a\varphi^a - dA'\partial_r h - (d-1)\partial_r H - 8V\nu - e^{-2A}\square h = 0, \quad (\text{B4})$$

whereas the mixed component gives

$$(d-1)A'\nu - \frac{1}{2}\partial_r h - 2\bar{\Phi}'_a\varphi^a = 0. \quad (\text{B5})$$

From the tangential component of Einstein's equations, we obtain

$$\begin{aligned} \frac{\partial_r^2 h}{2(d-1)} + \frac{dA'}{d-1}\partial_r h + \frac{A'}{2}\partial_r H - A'\partial_r \nu + \frac{e^{-2A}}{2(d-1)}\square h + \frac{8V}{d-1}\nu + \frac{4V_a\varphi^a}{d-1} = \\ -\frac{2}{d-1}\sum_i \delta_i(\lambda_{(i)}\nu + \partial_a\lambda_{(i)}\varphi^a), \end{aligned} \quad (\text{B6})$$

and

$$\frac{1}{2}\partial_r^2 H + \frac{dA'}{2}\partial_r H + e^{-2A}\square\nu + \frac{d-2}{2(d-1)}e^{-2A}\square h = 0. \quad (\text{B7})$$

Equations (B3), (B6), and (B7) lead to the boundary conditions

$$\varphi'^a \Big|_{r_i} = \bar{\Phi}'^a \nu + \partial_c \lambda_{(i)}^a \varphi^c \Big|_{r_i}, \quad (\text{B8})$$

$$\frac{\partial_r h}{2(d-1)} - A'\nu + \frac{2}{d-1}\bar{\Phi}'_a\varphi^a \Big|_{r_i} = 0, \quad (\text{B9})$$

and

$$\partial_r H \Big|_{r_i} = 0. \quad (\text{B10})$$

Equation (B8) gives the boundary condition for the scalar fluctuations. In the special case of one scalar, this expression agrees with the one given in [24]. (B9) is actually implied by equation (B5) and therefore does not give us any new information. Finally, the boundary condition for the variable H shows that we may put

$$\mathbf{c}_2 \equiv e^{-2A}\partial_\mu\nu^\mu - \frac{1}{2}\partial_r H \Big|_{r_i} = 0 \quad (\text{B11})$$

at the boundary. The reason is that since the form of the boundary conditions must obey 4d gauge invariance, the expression $H' = 0$ must generalize to $\mathbf{c}_2 = 0$ had we included the ν^μ fluctuations as well (this can be checked explicitly).

Appendix C: Translation to Gauge Invariant Variables

The significance of that one of the boundary conditions (B11) reads $\mathbf{c}_2 = 0$ is that using this relation, and the fact that \mathbf{c}_2 can be gauged away in the bulk, one now finds a one-to-one map between the gauge-invariant variables $(\mathbf{a}^a, \mathbf{b}, \mathbf{c})$ (defined in (25)) and the fluctuations (φ^a, ν, h) . We have that

$$\begin{aligned} h &= -2(d-1)A'e^{2A}\square^{-1}\mathbf{c}, \\ \varphi^a &= \mathbf{a}^a - \bar{\Phi}'^a e^{2A}\square^{-1}\mathbf{c}, \\ \nu &= \mathbf{b} - e^{2A}\square^{-1}(2A'\mathbf{c} + \partial_r\mathbf{c}). \end{aligned} \quad (\text{C1})$$

Let us proceed to derive expressions for the equations of motion in the bulk and the boundary conditions in terms of $(\mathbf{a}^a, \mathbf{b}, \mathbf{c})$. The boundary condition for the scalar fluctuations (B8) becomes

$$\mathcal{D}_r\mathbf{a}^a \Big|_{r_i} = \bar{\Phi}'^a \mathbf{b} + \left(V^a - dA'\bar{\Phi}'^a - \lambda^a|_c \bar{\Phi}'^c \right) e^{2A}\square^{-1}\mathbf{c} + \lambda^a|_c \mathbf{a}^c \Big|_{r_i}. \quad (\text{C2})$$

For the equation of motion for the scalar fluctuations in the bulk (B3), we obtain

$$\left[\mathcal{D}_r^2 + dA'\mathcal{D}_r + e^{-2A}\square\right]\mathbf{a}^a - \left[V_{|c}^a - \mathcal{R}_{bcd}^a \bar{\Phi}'^b \bar{\Phi}'^d\right]\mathbf{a}^c - \bar{\Phi}'^a(\mathbf{c} + \partial_r \mathbf{b}) - 2V^a \mathbf{b} = 0, \quad (\text{C3})$$

whereas the linearized Einstein's equations (B4), (B5), and (B7) lead to

$$\mathbf{c} = -\frac{2}{(d-1)A'} \left(\bar{\Phi}'_a \mathcal{D}_r - \frac{4V\bar{\Phi}'_a}{(d-1)A'} - V_a \right) \mathbf{a}^a, \quad (\text{C4})$$

$$\mathbf{b} = \frac{2\bar{\Phi}'_a \mathbf{a}^a}{(d-1)A'}, \quad (\text{C5})$$

and

$$\partial_r \mathbf{c} + dA' \mathbf{c} - e^{-2A} \square \mathbf{b} = 0, \quad (\text{C6})$$

respectively (where we have used the latter two equations in deriving the first). We recognize these equations from [25] and [27]. Solving for \mathbf{b} and \mathbf{c} in (C4) and (C5), we can now rewrite (C2) and (C3) in terms of only the scalar fluctuations \mathbf{a}^a . We obtain that \mathbf{a}^a satisfies the following equation of motion in the bulk

$$\left[\mathcal{D}_r^2 + dA'\mathcal{D}_r + e^{-2A}\square\right]\mathbf{a}^a - \left[V_{|c}^a - \mathcal{R}_{bcd}^a \bar{\Phi}'^b \bar{\Phi}'^d + \frac{4(\bar{\Phi}'^a V_c + V^a \bar{\Phi}'_c)}{(d-1)A'} + \frac{16V\bar{\Phi}'^a \bar{\Phi}'_c}{(d-1)^2 A'^2}\right]\mathbf{a}^c = 0, \quad (\text{C7})$$

with boundary conditions

$$\begin{aligned} & \left[\delta_b^a + e^{2A} \square^{-1} \left(V^a - dA' \bar{\Phi}'^a - \lambda_{|c}^a \bar{\Phi}'^c \right) \frac{2\bar{\Phi}'_b}{(d-1)A'} \right] \mathcal{D}_r \mathbf{a}^b \Big|_{r_i} = \\ & \left[\lambda_{|b}^a + \frac{2\bar{\Phi}'^a \bar{\Phi}'_b}{(d-1)A'} + e^{2A} \square^{-1} \frac{2}{(d-1)A'} \left(V^a - dA' \bar{\Phi}'^a - \lambda_{|c}^a \bar{\Phi}'^c \right) \left(\frac{4V\bar{\Phi}'_b}{(d-1)A'} + V_b \right) \right] \mathbf{a}^b \Big|_{r_i}. \end{aligned} \quad (\text{C8})$$

In the special case where there is a superpotential W , these expressions can be written as

$$\left[\left(\delta_b^a \mathcal{D}_r + W_{|b}^a - \frac{W^a W_b}{W} - \frac{2d}{d-1} W \delta_b^a \right) \left(\delta_c^b \mathcal{D}_r - W_{|c}^b + \frac{W^b W_c}{W} \right) + \delta_c^a e^{-2A} \square \right] \mathbf{a}^c = 0, \quad (\text{C9})$$

and

$$\begin{aligned} & \left[\delta_b^a + e^{2A} \square^{-1} \left(\lambda_{|c}^a - W_{|c}^a \right) \frac{W^c W_b}{W} \right] \mathcal{D}_r \mathbf{a}^b \Big|_{r_i} = \\ & \left[\lambda_{|b}^a - \frac{W^a W_b}{W} + e^{2A} \square^{-1} \left(\lambda_{|c}^a - W_{|c}^a \right) \frac{W^c W_d}{W} \left(W_{|b}^d - \frac{W^d W_b}{W} \right) \right] \mathbf{a}^b \Big|_{r_i}. \end{aligned} \quad (\text{C10})$$

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